PLANE DEPTH SEQUENCING

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Abstract.

1. Definitions

Definition 1.1. Let G be a graph with a plane embedding, and let C be the outer cycle of that embedding. The *plane depth* of a vertex $v \in V(G)$ relative to the embedding and C is

$$\mathrm{depth}(v) = \min_{u \in V(C)} d(v, u),$$

where $d(v, u)$ denotes the graph distance between v and u in G.

Definition 1.2. An edge $\{u, v\} \in E(G)$ is a *level edge* if $\mathrm{depth}(u) = \mathrm{depth}(v)$.

Definition 1.3. Let G be a maximal planar graph with a plane embedding and outer cycle C. The *deep embedding* of G is the graph G' obtained from G by the following operation: for every 3-cycle $\{u, v, w\} \subseteq V(G)$ such that

$$\mathrm{depth}(u) = \mathrm{depth}(v) = \mathrm{depth}(w),$$

add a new vertex x to G adjacent to each of u, v, and w.

Lemma 1.4. *Let G' be the deep embedding of a maximal planar graph G. For each face of G', the plane depths of its three vertices are either $d, d+1, d+1$ or $d, d, d+1$ for some $d \geq 0$.*

Proof. We first establish that for any edge $\{p, q\}$ in G, the depths of p and q differ by at most 1. Suppose for contradiction that $\mathrm{depth}(p) = d$ and $\mathrm{depth}(q) = d + n$ for some $n \geq 2$. Since $\mathrm{depth}(p) = d$, there exists a path of length d from p to some vertex of C. Prepending the edge $\{q, p\}$ gives a path of length $d + 1$ from q to C, so $\mathrm{depth}(q) \leq d + 1 < d + n$, a contradiction. The case $\mathrm{depth}(q) = d - n$ is handled identically: there exists a path of length $d - n$ from q to some vertex of C, and prepending the edge $\{p, q\}$ gives a path of length $d - n + 1 \leq d - 1 < d$ from p to C, contradicting $\mathrm{depth}(p) = d$.

Since G is a triangulation, every interior face of G is a triangle $\{u, v, w\}$ with all three pairs adjacent. By the above, each pair of vertices in a triangle differs in depth by at most 1, so no triangle can contain vertices of depths d and $d + 2$ simultaneously. The possible depth patterns for a triangle in G are therefore exactly d, d, d, or $d, d, d + 1$, or $d, d + 1, d + 1$.

We now consider each case under the deep embedding.

Case 1: depths $d, d, d + 1$ or $d, d + 1, d + 1$. These triangles are not modified by the deep embedding, so they remain as faces of G' with the stated depth patterns, satisfying the lemma.

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Case 2: depths d, d, d. The deep embedding inserts a new vertex x adjacent to u, v, and w, replacing the face $\{u, v, w\}$ with three new faces $\{u, v, x\}$, $\{v, w, x\}$, and $\{u, w, x\}$. It remains to determine the depth of x in G'. Since x is adjacent only to u, v, and w, every path in G' from x to C must pass through one of them, so x has strictly greater depth than u, v, and w. Each of the three new faces thus has depth pattern $d, d, d + 1$, satisfying the lemma.

Since every face of G' falls into one of these cases, the result follows. □