LEVEL RESOLUTIONS OF MAXIMAL PLANAR GRAPHS

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Abstract. We propose a structural reformulation of the four color theorem in terms of *level resolutions* of maximal planar graphs. A level structure on a plane graph G is defined by BFS from a chosen level source (either a face or a degree-3 vertex), partitioning vertices into levels. A triangulation G' on the same vertex set is a *level resolution* of G from this source if the subgraphs of G' induced by even- and odd-level vertices are both bipartite. By construction, any level resolution admits an explicit 4-coloring obtained by 2-coloring each parity subgraph independently. The structural foundation of this approach is that each level subgraph L_k of G is outerplanar, and outerplanar graphs are 3-chromatic; the level-resolution problem is precisely to flip edges of G to reduce each L_k from chromatic number 3 to 2. We present computational results characterizing which isomorphism classes of maximal planar graphs on $n = 6, \ldots, 11$ vertices arise as level resolutions, and verify that every iso-class is reachable at every tested size.

1. Introduction

The four color theorem (4CT) asserts that every planar graph is 4-colorable. Equivalently, every maximal planar graph (triangulation) is 4-colorable. The Appel–Haken proof [1] and subsequent Robertson–Sanders–Seymour–Thomas refinement [2] rely on discharging arguments and computer-verified reducible configurations. Human-readable proofs remain elusive.

We propose a different structural approach. Given a plane triangulation G and a choice of *level source*, BFS from the source partitions the vertices into levels. A triangulation G' on the same vertex set is a *level resolution* of G if, when its vertices are labelled by the parity of their G-levels, the subgraph of G' induced by even-parity vertices and the subgraph induced by odd-parity vertices are both bipartite. The 4-coloring of G' then follows by definition: 2-color each parity subgraph and identify the four resulting classes with four distinct colors.

The remaining question is when level resolutions exist. We conjecture:

(i) every plane triangulation G' is a level resolution of some plane triangulation G via some level source; or, in a restricted form,
(ii) every plane triangulation of minimum degree at least 4 is a level resolution of some plane triangulation.

This paper formalizes the definitions and presents computational evidence bearing on (i)–(ii) for small vertex counts.

2010 *Mathematics Subject Classification.* Primary .

2. Definitions

Throughout, $G = (V, E)$ is a plane maximal planar graph (a triangulation) with a fixed planar embedding Π_G. We write $|V| = n$, so $|E| = 3n - 6$ and G has $2n - 4$ triangular faces.

Definition 2.1 (Level source)**.** A *level source* of G is either:

- a face F of G (all vertices of F are level-0 sources), or
- a vertex v of degree 3 (the singleton $\{v\}$ is a level-0 source).

Definition 2.2 (Levels)**.** Given a level source $S \subseteq V$, the *level* of $v \in V$ is $\ell_G(v) = \mathrm{dist}_G(v, S)$, the graph distance from v to the nearest source vertex.

Definition 2.3 (Parity subgraph)**.** Let G be a triangulation with level source S, and let G' be a triangulation on the same vertex set as G. The *even parity subgraph* $E_{G,S}(G')$ is the subgraph of G' induced by $\{v \in V : \ell_G(v) \equiv 0 \pmod 2\}$. The *odd parity subgraph* is defined analogously for odd ℓ_G.

Definition 2.4 (Level resolution)**.** A triangulation G' on the same vertex set as G is a *level resolution* of G from level source S if both the even and odd parity subgraphs $E_{G,S}(G')$ and $O_{G,S}(G')$ are bipartite.

By construction, when G' is a level resolution of G via S, an explicit proper 4-coloring of G' is obtained by 2-coloring each parity subgraph independently (e.g., via BFS) and assigning the four resulting classes to distinct colors: even vertices receive red/blue, odd vertices receive yellow/green. The edges of G' partition into (i) edges within a parity subgraph, properly colored by the bipartition of that subgraph; and (ii) edges between an even-parity and odd-parity vertex, which connect disjoint color sets and so are properly colored.

3. Structural foundation: outerplanarity of level subgraphs

For each integer $k \geq 0$ and each (G, S), write L_k for the subgraph of G induced by the level-k vertices.

Theorem 3.1. *For every plane triangulation G and every level source S of G, each level subgraph L_k is outerplanar.*

Proof. For $k = 0$, L_0 is either a single vertex (when S is a degree-3 vertex) or the triangle bounding the source face (when S is a face), both outerplanar. Fix $k \geq 1$ and suppose, for contradiction, that L_k is not outerplanar.

Let D_k denote the planar drawing of L_k inherited from Π_G: that is, the set of points and curves in the plane representing the vertices and edges of L_k exactly as they appear in the embedding Π_G. Since L_k is not outerplanar, no face of D_k has every vertex of L_k on its boundary.

Let F^* be the face of D_k containing the source: when $S = \{v\}$, the face containing the point v; when S is a face F of G, the face containing the open region of F together with its three bounding vertices. The latter is well defined because each vertex of F lies at level 0 (hence is not a vertex of L_k) and each edge of F joins two level-0 vertices (hence is not an edge of L_k), so F and its boundary lie in a single component of $\mathbb{R}^2 \setminus D_k$. By assumption there exists $u \in L_k$ with $u \notin \partial F^*$.

Choose a BFS path $P : v_0, v_1, \ldots, v_k = u$ with $v_0 \in S$ and $v_i \in L_i$. For $0 \leq i \leq k-1$, v_i lies in L_i and so is not a vertex of L_k; for $1 \leq i \leq k$, the edge

$v_{i-1}v_i$ joins L_{i-1} to L_i and so is not an edge of L_k. Hence, viewed as a curve in the plane, P meets the drawing D_k only at its endpoint u.

The complement $\mathbb{R}^2 \setminus D_k$ is open, and $P \setminus \{u\}$ is its continuous image of a connected set, hence lies in a single face of D_k. Since $v_0 \in F^*$, in fact $P \setminus \{u\} \subseteq F^*$, so $u \in \overline{F^*}$ and therefore $u \in \partial F^*$, contradicting the choice of u. □

The combinatorial significance of Theorem 3.1 is that outerplanar graphs are 3-chromatic [4]: their chromatic number is at most 3. Hence each L_k admits an independent 3-coloring, giving an immediate (but suboptimal) coloring of G using at most $3 \cdot \mathrm{depth}(G, S)$ colors when levels are colored independently. To recover a 4-coloring of G' via the parity-2-coloring strategy, what is required is to reduce each L_k's chromatic number from 3 to 2, equivalently to remove every odd cycle from each L_k:

Proposition 3.2. *If G' is a triangulation on the same vertex set as G such that for every k, the subgraph of G' induced by the level-k vertices of (G, S) is bipartite, and G' contains no edge between vertices at G-levels of equal parity and differing by exactly 2, then G' is a level resolution of G via S.*

Proof. The even parity subgraph $E_{G,S}(G')$ is the disjoint union of the even-level subgraphs of G' (since by hypothesis no edge of G' joins two even levels), each of which is bipartite. A disjoint union of bipartite graphs is bipartite. The same argument applies to the odd parity subgraph. □

This is the form of level resolution we seek to realize constructively: flips applied to G that break every odd cycle in every L_k without introducing cross-parity edges of distance 2.

4. The four-color conjecture via level resolutions

Conjecture 4.1 (Resolution preimage)**.** Every plane triangulation G' on n vertices is a level resolution of some plane triangulation G on n vertices.

If Conjecture 4.1 holds, the 4-coloring of any triangulation G' follows from the definition: exhibit a level-resolution preimage G, compute the BFS levels in G from the witness source, and 4-color G' via the parity 2-coloring.

5. Computational evidence

We enumerated all non-isomorphic triangulations on $n \in \{6, \ldots, 11\}$ via vertex insertion followed by edge-flip closure (see `triangulation_gen.py` and the faster `triangulation_gen_fast.py` for $n \geq 11$). For each isomorphism class, we computed the full set of iso-classes reachable as level resolutions across all valid level sources.

5.1. Coverage at $n = 6, \ldots, 11$.

Table 1 lists the resolution behavior for each iso-class. A class T_i is *covered* if it appears as the resolution iso-class of some triangulation.

Observation 5.1. For every $n \in \{6, \ldots, 11\}$, every plane-triangulation iso-class on n vertices is a level resolution of some plane triangulation on the same vertex set.

n	Iso-classes	Reachable as level resolutions
6	2	all 2
7	5	all 5
8	14	all 14
9	50	all 50
10	233	all 233
11	1249	all 1249

Table 1. Iso-class coverage under the level-resolution definition.

Equivalence to 4-colorability. A 2-partition $V = V_0 \sqcup V_1$ for which both $G'[V_0]$ and $G'[V_1]$ are bipartite induces a proper 4-coloring of G' (combine the bipartition of V_0 into colors $\{R, B\}$ and that of V_1 into $\{Y, G\}$), and conversely, any proper 4-coloring grouped pairwise produces such a partition. Hence by Definition 2.4, G' is a level resolution of some (G, S) if and only if G' admits a bipartite 2-partition of cardinality realizable as $(|V_e|, |V_o|)$ for some level source. Surjectivity at a given n is therefore equivalent to 4-colorability of every triangulation on n vertices together with realizability of the partition cardinality by some BFS. Our computational verification of Observation 5.1 does not invoke 4CT: we enumerate vertex partitions directly and check bipartiteness of the induced subgraphs.

5.2. **Surjectivity at $n = 12$: the icosahedron.** The icosahedron is the unique 5-regular triangulation on 12 vertices and a natural test case at $n = 12$ since it has no degree-3 vertex (so the md_4 restriction is irrelevant) and high symmetry constrains the achievable parity-cardinality splits to $(6, 6)$ from any source. We verify directly that the icosahedron admits a bipartite 2-partition of cardinality $(6, 6)$: with vertices labelled as in the standard icosahedral graph, the partition $\{0, 1, 2, 3, 4, 7\} \mid \{5, 6, 8, 9, 10, 11\}$ has both classes inducing bipartite subgraphs (each is a 6-cycle). By Definition 2.4, the icosahedron is therefore a level resolution of some plane triangulation on 12 vertices.

Observation 5.2. The icosahedron is a level resolution of some plane triangulation on 12 vertices.

5.3. **Restatement of the resolution-preimage conjecture.** In light of Observations 5.1 and 5.2, we restate Conjecture 4.1 more confidently:

Conjecture 5.3 (md_4 surjectivity)**.** For every $n \geq 6$, every minimum-degree-4 plane triangulation on n vertices is a level resolution of some plane triangulation on n vertices.

By the equivalence noted in Section 3, this is equivalent to a 4-coloring statement: every minimum-degree-4 plane triangulation admits a proper 4-coloring whose color-class cardinalities, grouped pairwise, match some BFS-level parity cardinality on the same vertex set. Since the unrestricted preimage conjecture also appears to hold at every tested n, the md_4 restriction may be unnecessary; we retain it here as the form most amenable to the constructive techniques explored in Section 6.

6. An edge-flip resolution algorithm

We describe an iterative edge-flip procedure aimed at producing, for a given (G, S), a triangulation G' on the same vertex set whose simple level cycles (with respect to the G-levels from S) are all even.

6.1. Apex classification of L_k-edges.

Let $k \geq 1$. For each $uv \in E(L_k)$, the two triangles of G bounding uv have third vertices w, x, called the *apexes* of uv, with $\ell_G(w), \ell_G(x) \in \{k-1, k, k+1\}$ by BFS. We call uv *intra-level* when $\ell_G(w) = \ell_G(x) = k$, and *cross-level* otherwise.

Lemma 6.1. *If both apexes of $uv \in E(L_k)$ are at level $k-1$, then uv is a bridge of L_k.*

Proof sketch. In a plane triangulation, the neighbors of u in G at level $\leq k-1$ form a contiguous arc in the cyclic order around u. If both apexes w, x of uv lie at level $k-1$ on opposite sides of uv, then v lies in the complementary cyclic arc, which contains no other level-k neighbor of u. The symmetric statement around v gives that u is v's only level-k neighbor in the corresponding arc, so uv is a bridge of L_k. □

In particular every edge on a cycle of L_k has at least one apex at level k or $k+1$.

Proposition 6.2. *Flipping $uv \in E(L_k)$ with apexes w, x replaces uv with wx in G. The new edge wx belongs to L_k iff $\ell_G(w) = \ell_G(x) = k$, and to L_{k+1} iff $\ell_G(w) = \ell_G(x) = k+1$; otherwise wx is cross-parity and lies in no level subgraph. In all cases uv is removed from L_k.*

6.2. Cross-level flip pass.

For each odd simple cycle C of each L_k containing a cross-level edge, flip one such edge. By Proposition 6.2 the new edge either enters L_{k+1} (in the apex case $(k+1, k+1)$) or is cross-parity (otherwise). Choosing apex pairs distinctly across cycles makes the set of new edges entering any single level L_j a matching, hence a forest, and similarly for new same-parity-distance-2 edges entering the relevant parity subgraph; these therefore introduce no odd cycle.

6.3. Tricky-everywhere cycles.

After the cross-level pass, the only odd simple cycles remaining in any L_k are those whose every edge is intra-level; we call such a cycle *tricky-everywhere*. By Proposition 6.2, flipping any edge of a tricky-everywhere cycle replaces it with another edge of L_k, so the local triangle pair (uvw, uvx) becomes (uwx, vwx): still a pair of odd triangles inside L_k. To make global progress on these cycles we use a facial-depth potential to choose the flip.

Definition 6.3 (Facial depth)**.** Let L_k be drawn with the outerplanar embedding inherited from Π_G, let D be the dual graph of this drawing with the outer face removed, and let $\mathcal{B}$ be the set of inner faces incident to at least two edges of the outer face of L_k. The *facial depth* of an inner face F of L_k is

$$\operatorname{depth}(F) \;=\; \min_{F' \in \mathcal{B}} \operatorname{dist}_D(F, F'),$$

with the convention $\operatorname{depth}(F) = \infty$ if no such F' exists.

6.4. **The algorithm.**

(1) *Cross-level flip pass.* For each L_k and each odd simple cycle $C \subseteq L_k$ containing a cross-level edge, flip one such edge, selecting apex pairs to keep the newly added edges a matching in each target level subgraph and in the relevant parity subgraph.
(2) *Intra-level flip loop.* While some L_k contains a tricky-everywhere odd simple cycle:
   (a) compute facial depths for all simple level cycles of L_k;
   (b) among tricky-everywhere odd simple cycles of maximum facial depth, pick any C;
   (c) among the edges of C, pick one whose other incident inner face has minimum facial depth, and flip it.

Observation 6.4. For every plane triangulation G on $n \in \{9, 10, 11\}$ vertices, every level source S, and every k such that L_k contains a tricky-everywhere odd simple cycle, Step 2 terminates with no tricky-everywhere odd simple cycle remaining in any L_k. Moreover, the total number of tricky-everywhere odd simple cycles strictly decreases on every flip chosen by Step 2(c).

Question 6.5. Does Observation 6.4 hold for all n? Equivalently, does the count of tricky-everywhere odd simple cycles strictly decrease on every Step 2(c) flip, in every plane triangulation?

7. Discussion and open questions

The computational results suggest the following:

(1) Conjecture 4.1 (resolution preimage) holds at every tested size: all isoclasses on $n \in \{6, \ldots, 11\}$ vertices arise as level resolutions, and the icosahedron does at $n = 12$ (Observations 5.1 and 5.2).
(2) Each level subgraph L_k of G is outerplanar (Theorem 3.1), so each L_k is 3-chromatic classically and independently of 4CT. The level-resolution problem reduces to flipping edges of G so that each L_k's chromatic number drops from 3 to 2, while avoiding creation of G-level-2 same-parity edges (Proposition 3.2).
(3) Under Definition 2.4, being a level resolution is equivalent to admitting a proper 4-coloring whose color cardinalities group pairwise to a BFS-realizable parity split. The structural framing through outerplanarity refines this: a constructive 4-coloring of G' is obtained via independent 2-colorings of each L_k in G', and the proof obligation is purely about removing odd cycles within outerplanar graphs by local edge flips, an operation that does not invoke 4CT.

The algorithm of Section 6 is the candidate constructive answer: cross-level flips dispose of every odd cycle of L_k that admits one, and the facial-depth-guided intra-level flip loop attacks the residual tricky-everywhere cycles. Observation 6.4 records that the loop terminates on all tested (G, S, k) at $n \le 11$; Question 6.5 asks whether termination holds for all n.

8. Implementation

The code accompanying this paper consists of the following modules:

- `level_cycles.py`: core library for levels, level cycles, flip candidates, and resolution enumeration.
- `triangulation_gen.py`: enumeration of all non-isomorphic triangulations on n vertices via vertex-insertion plus flip closure.
- `coverage.py`: iso-class coverage reports with optional source and target filters.
- `balanced_layout.py`: a planar drawing routine that starts from a Tutte embedding and uses random-search optimization to equalize interior face areas while maintaining planarity.
- `four_color.py`: 4-coloring of G' via independent BFS 2-colorings of parity subgraphs.
- Visualization scripts: `plot_oct.py`, `n7_examples.py`, `four_color_viz.py`.

References

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