THE MEDIAL PIGEONHOLE PROGRAMME

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Abstract. Building on the medial tire decomposition of a plane triangulation, we formulate a pigeonhole programme for the Four Colour Theorem in medial terms. Each tire carries a boundary-state restriction relation, and a proper vertex 3-colouring of the full medial graph is a compatible selection of these boundary states across the tire tree. We state a chain-pigeonhole conjecture asserting that the restriction relations cannot remain mutually disjoint along every branch, and we refine the boundary states by recording how two-colour Kempe cycles are routed through each annular tire region. This yields a Kempe-enhanced restriction relation and a notion of Kempe-compatible gluing along level cycles.

1. Introduction

This paper continues the medial tire programme begun in [1]. We use freely the terminology and notation introduced there. For a plane triangulation G with fixed embedding, $M(G)$ denotes the full medial graph, and the tire-tree decomposition $\mathcal{T}(G,S)$ at a level source S of [2] induces a decomposition of $M(G)$ into full medial tire graphs $\mathsf{M}(T)$, one for each tread T, glued along their boundary medial vertex sets $\partial_{\text{out}}\mathsf{M}(T)$ and $\partial_{\text{in}}\mathsf{M}(T)$. We also use the annular medial cycle $A(T)$, its up and down teeth and their apexes, the bites and the auxiliary plane graph $B(T)$, and the medial tire restriction relation R_T of [1].

By the Tait–medial correspondence of [1], proper vertex 3-colourings of $M(G)$ are in natural bijection with proper 3-edge-colourings of the cubic planar dual G^*. Thus the Four Colour Theorem is the assertion that the full medial graph of every plane triangulation is properly vertex 3-colourable, and the medial tire decomposition turns this into a question about how local boundary colourings compose across the tire tree.

2. A medial pigeonhole programme

The restriction relation R_T records exactly the local information needed to pass a medial 3-colouring through a tire. In a nested chain

$$T_0 \supset T_1 \supset \cdots \supset T_k,$$

the outer boundary state of T_{i+1} must match an inner boundary state allowed by R_{T_i}. Thus a proof of the Four Colour Theorem in this framework would follow from a structural reason that these restriction sets cannot remain mutually disjoint along every branch of the tire tree.

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Definition 2.1 (Medial boundary state)**.** A *medial boundary state* on a boundary set $\partial\mathsf{M}(T)$ is a proper vertex 3-colouring of the subgraph induced by that boundary set, considered up to permutation of the three colours and the dihedral symmetries of the boundary walk when that boundary is a cycle.

Conjecture 2.2 (Medial chain-pigeonhole principle)**.** *There is a function $N(k)$ such that the following holds. Let $T_0 \supset T_1 \supset \cdots \supset T_{N(k)}$ be a nested chain of tire treads whose relevant boundary medial walks have length at most k. Then two adjacent restriction relations in the chain have compatible medial boundary states after colour permutation and boundary symmetry. Equivalently, the chain contains a local gluing step that cannot be obstructed by disjoint proper vertex* 3*-colouring restrictions.*

Conjecture 2.3 (Medial tire route to the Four Colour Theorem)**.** *For every plane triangulation G and every level source S, the restriction relations $\{R_T : T \in V(\mathcal{T}(G,S))\}$ admit a compatible selection of boundary states across the tire tree. Hence $M(G)$ is properly vertex* 3*-colourable, G^* is properly* 3*-edge-colourable, and G is properly* 4*-vertex-colourable.*

Remark 2.4. Conjecture 2.3 is equivalent in strength to the Four Colour Theorem when combined with Tait's correspondence. The point of the formulation is not to weaken the target theorem, but to move the obstruction into finite boundary-state restrictions carried by annular medial tire pieces.

3. Kempe-cycle conservation across medial tires

We now record an additional structure carried by proper 3-colourings of medial graphs. This structure will be useful for describing how colourings glue across level cycles.

Let G be a plane triangulation and let $M = M(G)$ be its medial graph. Let

$$\varphi : V(M) \to \{1,2,3\}$$

be a proper 3-colouring of M. For a two-element colour set $P = \{a,b\} \subseteq \{1,2,3\}$, let M_P denote the subgraph of M induced by the vertices of colours a and b.

Since M is 4-regular and φ is proper, every vertex of M_P has degree 2 in M_P. Hence every component of M_P is a cycle. We call these components the P-Kempe cycles of φ.

Lemma 3.1 (Kempe chains are cycles)**.** *Let G be a plane triangulation, let $M = M(G)$, and let φ be a proper* 3*-colouring of M. For each $P \in \{\{1,2\},\{2,3\},\{3,1\}\}$, every component of M_P is a cycle.*

Proof. Let $v \in V(M_P)$. In the medial graph M, the vertex v has degree 4. Since φ is a proper 3-colouring, none of the neighbours of v has colour $\varphi(v)$. Thus all four neighbours of v have one of the two colours different from $\varphi(v)$.

In the medial graph of a plane triangulation, the neighbours of a medial vertex occur in two opposite pairs corresponding to the two faces incident with the corresponding edge of G. Around each such triangular face, the three medial vertices receive all three colours. Consequently, at v there are exactly two neighbours of each colour different from $\varphi(v)$. It follows that, in the subgraph induced by any two colours P, every vertex has degree 2. Hence each component of M_P is a cycle. □

Let T be a medial tire region. We regard T as an annular transition region whose boundary consists of one outer level cycle and finitely many inner level cycles:

$$\partial T = C_0 \sqcup C_1 \sqcup \cdots \sqcup C_m.$$

Here C_0 is the outer level cycle of T, and the cycles $C_1, \ldots, C_m$ are the inner level cycles. Each inner level cycle C_i is also the outer level cycle of the corresponding child region in the tire tree.

The following lemma is the basic conservation principle.

Lemma 3.2 (Kempe-cycle conservation across level cycles)**.** *Let C be a level cycle of M separating a parent side from a child side. Let K be a P-Kempe cycle for some $P \in \{\{1,2\},\{2,3\},\{3,1\}\}$. Then K cannot enter the child side of C without also leaving it.*

Equivalently, the incidences of K with C are paired by the components of K lying on the child side of C, and also paired by the components of K lying on the parent side of C.

Proof. By the preceding lemma, K is a cycle. The level cycle C separates the sphere into two closed regions, which we call the parent side and the child side. Consider the intersection of K with one of these regions. Since K is a cycle, no component of this intersection can have exactly one boundary endpoint on C. Each component is either closed within the region, or is a path with two boundary endpoints on C. Thus every entrance through C is paired with an exit through C. □

We now use these Kempe cycles to single out the colourings of a full medial tire graph that respect the annular tooth structure.

Definition 3.3 (Kempe-balanced colouring)**.** Let φ be a proper 3-colouring of the full medial tire graph $\mathsf{M}(T)$. For a colour pair $P = \{a, b\}$, let $\mathsf{M}(T)_P$ be the subgraph induced by the vertices of colours a and b. Since $\mathsf{M}(T)$ need not be 4-regular, the components of $\mathsf{M}(T)_P$ are paths or cycles; we call them the *P-Kempe chains* of φ. Every vertex of colour a or b lies on exactly one P-Kempe chain.

A *valid face* is the outer face of $\mathsf{M}(T)$, or an interior face of $B(T)$ that is not a tooth—namely the root face or a bite inner-gap face in the sense of [1]. The *tooth apexes incident to* a valid face F are:

- the up-tooth apexes ([1]), when F is the outer face;
- the singleton down-tooth apexes whose annular edge lies on F, when F is interior—the apex on annular edge m being incident to the innermost bite (i, j) with $i < m < j$, or to the root face if there is none.

Bite apexes are never incident to a valid face in this sense.

For a colour pair $P = \{a, b\}$ write $\nu_P(F)$ for the number of tooth apexes incident to F that are coloured a or b—equivalently, that lie on a P-Kempe chain. The colouring φ is *Kempe-balanced* if $\nu_P(F)$ is even for every valid face F and every colour pair P.

Remark 3.4 (Necessity of Kempe-balance). A proper 3-colouring of $\mathsf{M}(T)$ can be part of a proper 3-colouring of the whole medial graph $M(G)$ only when it is Kempe-balanced: if φ is the restriction to $\mathsf{M}(T)$ of a proper 3-colouring of $M(G)$, then φ is Kempe-balanced. Equivalently, a colouring of $\mathsf{M}(T)$ that fails the parity condition at some valid face and colour pair cannot extend to a proper 3-colouring of $M(G)$. This is an instance of Kempe-cycle conservation (Lemma 3.2). The tooth apexes incident to a valid face are boundary medial vertices ([1]) lying on a single level cycle of the tire decomposition: the up-tooth apexes lie on the outer level cycle, and the singleton down-tooth apexes incident to an interior non-tooth face lie on the inner level cycle bounding that face. In the 4-regular graph $M(G)$ each P-Kempe chain of $\mathsf{M}(T)$ closes up into a P-Kempe cycle, which by Lemma 3.2 meets each level cycle in an even number of P-coloured incidences; for a given valid face these incidences are exactly its incident tooth apexes coloured a or b, whence $\nu_P(F)$ is even.

This argument is verified computationally. For bite-free pieces—capped triangulated annuli on annular cycles of length $6, 8, 10, 12$—every proper 3-colouring of $M(G)$ restricts to a Kempe-balanced colouring. The same holds for pieces carrying a bite, including the case where singleton down teeth lie in the bite's inner-gap face: there the inner level cycle splits into a child level cycle per gap, and conservation across each child cycle supplies the parity (in the checked example the three singleton down apexes of a bite gap are a rainbow in every restriction).

More generally, let T be a medial tire region with boundary

$$\partial T = C_0 \sqcup C_1 \sqcup \cdots \sqcup C_m.$$

For a P-Kempe cycle K, every component of $K \cap T$ is either a cycle contained in T, or a path with two endpoints on ∂T. Thus the P-Kempe arcs inside T define a pairing of the P-coloured boundary incidences of

$$C_0 \sqcup C_1 \sqcup \cdots \sqcup C_m.$$

This motivates the following refinement of boundary states.

Definition 3.5 (Kempe-enhanced boundary state)**.** Let T be a medial tire region with outer level cycle C_0 and inner level cycles $C_1, \ldots, C_m$. Let

$$\mathcal{C}(T) = C_0 \sqcup C_1 \sqcup \cdots \sqcup C_m.$$

A *Kempe-enhanced boundary state* on T consists of the following data:

(1) a boundary colouring
$$\alpha : V(\mathcal{C}(T)) \to \{1, 2, 3\};$$
(2) for each colour pair
$$P \in \{\{1, 2\}, \{2, 3\}, \{3, 1\}\},$$
a pairing π_P of the P-coloured boundary incidences of $\mathcal{C}(T)$ induced by the P-Kempe arcs lying inside T.

We write such a state as

$$\kappa = (\alpha, \pi_{12}, \pi_{23}, \pi_{31}).$$

Given a proper 3-colouring φ of the medial tire graph $M(T)$, the restriction of φ to the boundary level cycles gives the boundary colouring α, while the two-colour Kempe arcs inside T give the pairings $\pi_{12}, \pi_{23}, \pi_{31}$. Thus φ determines a Kempe-enhanced boundary state, denoted

$$\kappa_T(\varphi).$$

Definition 3.6 (Kempe-enhanced restriction relation)**.** The *Kempe-enhanced restriction relation* of T is

$$\mathcal{K}_T = \{\kappa_T(\varphi) : \varphi \text{ is a proper 3-colouring of } M(T)\}.$$

This refines the ordinary boundary-colouring relation by recording not only which boundary colourings extend across T, but also how the two-colour Kempe cycles are routed through the annular tire region.

The annular structure of a tire is useful in two distinct ways. First, it gives a bounded transition region between level cycles: the colouring of the annular medial cycle controls, and in many cases determines, the colouring of the remaining medial tire vertices. Thus the number of possible transition states is bounded in terms of the annular structure, rather than the total size of the subtree below the tire. Second, it describes how the outer level cycle and the inner level cycles are related by Kempe arcs. The level cycles are the gluing interfaces, while the annular tire is the transition operator between them.

Definition 3.7 (Kempe-compatible gluing)**.** Let T be a medial tire region and let U be a child region glued to T along a common level cycle C. Thus C is an inner level cycle of T and the outer level cycle of U.

Let

$$\kappa_T = (\alpha_T, \pi_{12}^T, \pi_{23}^T, \pi_{31}^T) \in \mathcal{K}_T$$

and

$$\kappa_U = (\alpha_U, \pi_{12}^U, \pi_{23}^U, \pi_{31}^U) \in \mathcal{K}_U.$$

We say that κ_T and κ_U are *Kempe-compatible along* C if:

(1) the boundary colourings agree on C:

$$\alpha_T|_{V(C)} = \alpha_U|_{V(C)};$$

(2) for each colour pair

$$P \in \{\{1,2\}, \{2,3\}, \{3,1\}\},$$

the pairings π_P^T and π_P^U compose along the P-coloured incidences of C without producing an unpaired endpoint.

When these conditions hold, the composed pairings determine a Kempe-enhanced boundary state on the exposed boundary of $T \cup_C U$.

In these terms, gluing local colourings is not merely a matter of matching boundary colours. The colourings must also route their two-colour Kempe arcs compatibly across every shared level cycle. The ordinary restriction relation records whether a boundary colouring can be extended locally; the Kempe-enhanced relation additionally records the conservation of Kempe-cycle flow through the annular transition region.

For a tire with one outer level cycle and several inner level cycles,

$$\partial T = C_0 \sqcup C_1 \sqcup \cdots \sqcup C_m,$$

the parent tire may correlate the boundary states on the different inner cycles. The Kempe-enhanced relation records this correlation as a system of pairings among the P-coloured incidences of all boundary level cycles simultaneously. Thus one should view a medial tire as a multi-output transition operator

$$\mathcal{K}_T : C_0 \rightsquigarrow (C_1, \ldots, C_m),$$

rather than as an independent collection of binary transitions.

The guiding principle is therefore:

Level cycles are the interfaces used for gluing, while annular tire regions are the bounded transition regions that route Kempe cycles between those interfaces.

References

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