

# HEAWOOD RESTRICTIONS ON NESTED TIRE GRAPH DUALS

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ABSTRACT.

## 1. INTRODUCTION

A classical theorem of Tait recasts the Four Colour Theorem in dual, edge-colouring terms: a plane triangulation  $G$  is properly 4-vertex-colourable if and only if its dual cubic graph  $G'$  is properly 3-edge-colourable. Thus a minimal counterexample to the Four Colour Theorem – a smallest triangulation admitting no proper 4-colouring – corresponds to a smallest cubic plane graph admitting no proper 3-edge-colouring.

This paper continues the series studying that structure through the lens of *nested level duals*. The foundational vocabulary — level sources, levels, the inner planar dual  $G'$  and its dual depth, and tire graphs — is developed in the companion paper [3]; we refer to that paper for those definitions and rely on them throughout. In particular we use, without restating, the notions of:

- *level source*  $S$  and  $G$ -vertex levels  $\ell_G(v)$ ;
- the inner planar dual  $G'$  ([3, Definition 1.3]);
- *dual depth*  $\delta_G(d_f)$  ([3, Definition 1.4]);
- *tire graph*  $T = (B_{\text{out}}, O, E_{\text{ann}})$  with outer/inner boundaries and annular edges ([3, Definition 1.5]);
- the *tire-component lemma* ([3, Lemma 1.8]); and
- the *tire-tread partition theorem* ([3, Theorem 1.9]).

Throughout,  $G = (V, E)$  is a plane maximal planar graph (a triangulation) with a fixed planar embedding  $\Pi_G$ . We write  $|V| = n$ , so  $|E| = 3n - 6$  and  $G$  has  $2n - 4$  triangular faces.

The classical input is Heawood's face-sum identity [1]: for any proper 3-edge-colouring of a cubic plane graph  $H$ , assigning each face of  $H$  a number in  $\{+1, -1\}$  can be done so that the labels around every vertex of  $H$  sum to 0 (mod 3). In the triangulation  $G$  dual to  $H$  this becomes a  $\{+1, -1\}$  labelling of the *faces* of  $G$  whose incident-face sum at every vertex of  $G$  vanishes mod 3. Our aim is to record what this restriction forces along the boundary cycles of a nested tire graph, and to formulate a chain-pigeonhole programme in this Heawood labelling parallel to the medial programme of [4].

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## 2. CONNECTED TIRE CLUSTERS

The tire treads at a fixed depth partition the depth- $d$  faces of  $G$  [3], but distinct depth- $d$  tires need not be vertex-disjoint: a single vertex of  $G$  may lie on the source-side boundary of several depth- $d$  tires at once (this occurs exactly when the depth- $d$  faces around that vertex are split into more than one arc by depth- $(d-1)$  faces). We organise the depth- $d$  tires by this sharing.

**Lemma 2.1** (Same-depth tires meet only in vertices). *Let  $T \neq T'$  be two distinct tire treads at the same depth  $d$  in  $\mathcal{T}(G, S)$ , arising from connected components  $C', C''$  of the depth- $d$  dual subgraph  $G'_d$ . Then  $T$  and  $T'$  share no edge of  $G$ ; any intersection  $V(T) \cap V(T')$  consists of isolated vertices.*

*Proof.* An edge  $e$  of  $G$  shared by two depth- $d$  annular faces  $f_1, f_2$  is, by definition of the inner dual [3, Definition 1.3], a dual edge of  $G'$  joining  $d_{f_1}$  and  $d_{f_2}$ ; since  $\delta(d_{f_1}) = \delta(d_{f_2}) = d$ , this edge lies in  $G'_d$ , so  $d_{f_1}$  and  $d_{f_2}$  belong to the same component of  $G'_d$ . Hence no edge of  $G$  is shared by annular faces of two *different* components, and distinct depth- $d$  tires share no edge. Their intersection is therefore a set of isolated vertices.  $\square$

**Definition 2.2** (Connected tire cluster). Fix a nested tire decomposition  $\mathcal{T}(G, S)$  and a depth  $d$ . On the set of depth- $d$  tire treads define the relation

$$T \sim T' \iff V(T) \cap V(T') \neq \emptyset.$$

A *connected tire cluster* at depth  $d$  is the subgraph of  $G$

$$K = \bigcup_i T_i \subseteq G$$

obtained as the union (of underlying plane graphs) of the tires in a single connected component  $\{T_i\}$  of the transitive closure of  $\sim$ . A cluster consisting of a single tire is *trivial*; the connected tire clusters at depth  $d$  partition the depth- $d$  tires.

*Remark 2.3.* By Lemma 2.1 the constituent tires of a connected tire cluster are joined only at shared vertices, each of which is a cut vertex of  $K$ ; a connected tire cluster is thus a “cactus of tires” and is in general *not* itself a tire graph, since the annulus structure of [3, Definition 1.5] fails at each such pinch. The shared (cut) vertices are precisely the vertices that belong to more than one depth- $d$  tire.

A single vertex may belong to several tires at one depth — the high-degree case where its depth- $d$  faces split into many arcs — so the number of *tires* through a vertex is unbounded. Clustering collapses exactly this multiplicity: all tires through a vertex at a fixed depth share that vertex, hence lie in one cluster. The cluster count is therefore controlled.

**Proposition 2.4** (A vertex meets at most two clusters). *Every vertex  $v \in V(G)$  belongs to at most two connected tire clusters, namely at most one at each of the two consecutive depths  $\ell_G(v) - 1$  and  $\ell_G(v)$ . In particular a source vertex ( $\ell_G(v) = 0$ ) belongs to a single cluster.*

*Proof.* Write  $\ell = \ell_G(v)$ .

*Step 1:* every bounded face incident to  $v$  has dual depth  $\ell - 1$  or  $\ell$ . Let  $f$  be a bounded triangular face with  $v \in V(f)$ . Then  $\delta_G(d_f) = \min_{u \in V(f)} \ell_G(u) \leq \ell_G(v) = \ell$ . The other two vertices of  $f$  are adjacent to  $v$  in  $G$ , and the level function

$\ell_G(\cdot) = \text{dist}_G(\cdot, S)$  is 1-Lipschitz along edges, so each has level at least  $\ell - 1$ ; hence  $\delta_G(d_f) \geq \ell - 1$ . Thus  $\delta_G(d_f) \in \{\ell - 1, \ell\}$  (only  $\delta_G(d_f) = 0$  when  $\ell = 0$ ), so  $v$  bounds faces of, and therefore belongs to tires of, no depth other than  $\ell - 1$  or  $\ell$ .

*Step 2: at each depth, all tires through  $v$  lie in one cluster.* Fix  $d \in \{\ell - 1, \ell\}$  and let  $T, T'$  be depth- $d$  tires with  $v \in V(T) \cap V(T')$ . Then  $V(T) \cap V(T') \neq \emptyset$ , so  $T \sim T'$  in the sense of Definition 2.2, and all depth- $d$  tires containing  $v$  lie in a single connected component of  $\sim$  — one connected tire cluster  $K_d$ .

Combining the two steps,  $v$  belongs to at most the clusters  $K_{\ell-1}$  and  $K_\ell$ , i.e. to at most two connected tire clusters; when  $\ell = 0$  only  $K_0$  occurs.  $\square$

### 3. HEAWOOD RESTRICTIONS ON THE TIRE DUAL

We work inside a fixed nested tire decomposition  $\mathcal{T}(G, S)$  of  $G$  from a single-vertex level source  $S$  [3], and use the tire data  $T = (B_{\text{out}}, O, E_{\text{ann}})$  with annular faces  $F_{\text{ann}}$ , outer boundary  $B_{\text{out}}$ , and inner boundary  $B_{\text{in}}$  ([3, Definition 1.5]). Since  $O$  is outerplanar, every vertex of a tire lies on  $B_{\text{out}}$  or on the inner-boundary walk  $B_{\text{in}}$ ; a tire has no interior vertices.

**Definition 3.1** (Heawood face-labelling of a tire). A *Heawood face-labelling* of a tire graph  $T$  is a map

$$\lambda : F_{\text{ann}} \longrightarrow \{+1, -1\}$$

assigning a sign to each annular face of  $T$ . For a vertex  $v \in V(T)$ , write  $F_{\text{ann}}(v) \subseteq F_{\text{ann}}$  for the set of annular faces of  $T$  incident to  $v$ , and define the *induced vertex value*

$$\lambda^*(v) := \sum_{f \in F_{\text{ann}}(v)} \lambda(f) \pmod{3} \in \{0, 1, -1\}.$$

The value  $\lambda^*(v)$  is the *partial* face-sum at  $v$  taken over the annular faces of  $T$  alone, not over all faces of  $G$  incident to  $v$ .

*Remark 3.2.* Because a tire has no interior vertices, every annular face of  $T$  is incident to  $B_{\text{out}} \cup B_{\text{in}}$ , and a Heawood face-labelling is subject to *no* internal constraint: all  $2^{|F_{\text{ann}}|}$  sign assignments are admissible. The Heawood restriction is felt only on the two boundary cycles, through the induced vertex values  $\lambda^*$ .

**Definition 3.3** (Induced boundary sequences). Let  $\lambda$  be a Heawood face-labelling of  $T$ . Reading the vertices of  $B_{\text{out}}$  in clockwise order  $v_0, v_1, \dots, v_{p-1}$ , the *outer Heawood sequence* of  $(T, \lambda)$  is

$$\sigma_{\text{out}}(T, \lambda) := (\lambda^*(v_0), \dots, \lambda^*(v_{p-1})) \in \{0, 1, -1\}^p.$$

Reading the inner-boundary walk  $B_{\text{in}}$  in clockwise order  $w_0, \dots, w_{q-1}$  gives the *inner Heawood sequence*  $\sigma_{\text{in}}(T, \lambda) \in \{0, 1, -1\}^q$ . The *Heawood restriction relation* of  $T$  is the set

$$R_T := \{ (\sigma_{\text{out}}(T, \lambda), \sigma_{\text{in}}(T, \lambda)) : \lambda : F_{\text{ann}} \rightarrow \{+1, -1\} \}$$

of all (outer, inner) sequence pairs realisable by a single face-labelling, read up to rotation and the global sign-flip  $\lambda \mapsto -\lambda$  (equivalently  $\sigma \mapsto -\sigma$ ).

**Definition 3.4** (Heawood compatibility across an interface). Let  $T$  be a tire and  $T' \in \mathcal{T}(G, S)$  a child of  $T$ , so the outer boundary cycle  $B_{\text{out}}^{(T')}$  coincides with a bounded face of  $O^{(T)}$ ; let  $\gamma$  be this shared cycle, of length  $L$ , and let  $v$  range over

its vertices. Heawood face-labellings  $\lambda$  of  $T$  and  $\lambda'$  of  $T'$  are *compatible along  $\gamma$*  if at every shared vertex  $v$ ,

$$\lambda^*(v) + (\lambda')^*(v) \equiv 0 \pmod{3},$$

i.e. 0 is paired with 0 and +1 with  $-1$ . Equivalently, the inner Heawood sequence of  $T$  on  $\gamma$  is the pointwise negation mod 3 of the outer Heawood sequence of  $T'$  on  $\gamma$ , after reversing one of the two clockwise readings to account for the opposite rotational senses in which  $T$  and  $T'$  traverse  $\gamma$ .

*Remark 3.5.* Call  $v$  *interior* if it is not incident to the outer face of  $\Pi_G$ . For an interior vertex every incident face is bounded, and compatibility along  $\gamma$  at  $v$  is exactly the statement that the incident-face sum at  $v$  — over the parent's annular faces together with the child's — vanishes mod 3:

$$(3.1) \quad \sum_{f \ni v} \lambda(f) \equiv 0 \pmod{3} \quad \text{for every interior vertex } v \in V(G),$$

the sum ranging over the bounded faces incident to  $v$ . The interfaces of  $\mathcal{T}(G, S)$  are interior level cycles, so cluster compatibility only ever constrains interior vertices and is untouched by the outer face.

To pass from (3.1) to a colouring one must account for the outer face: an outer-boundary vertex is incident to the unbounded face  $f_\infty$ , whose label is omitted from the bounded sum. Extend  $\lambda$  by a single label  $\lambda(f_\infty) \in \{+1, -1\}$  on  $f_\infty$ . Then a family of Heawood face-labellings that is pairwise compatible along every interface of  $\mathcal{T}(G, S)$  assembles into a  $\{+1, -1\}$  labelling of *all* faces of  $G$  for which  $\sum_{f \ni v} \lambda(f) \equiv 0 \pmod{3}$  holds at every vertex — the outer-boundary vertices now carrying  $\lambda(f_\infty)$  in their sum. This is Heawood's face-sum identity [1] for a proper 3-edge-colouring of the full cubic dual of  $G$ , hence (by Tait) a proper 4-vertex-colouring of  $G$ .

**Why the programme runs between nested clusters.** The vanishing condition (3.1) at a vertex  $v$  is a constraint on the *full* face-star of  $v$ . To run a pigeonhole between two objects — a child and a parent — we need that full sum to split as exactly two one-sided contributions, so that each vertex label is the combination of a single child value and a single parent value. This is true at the level of connected tire clusters, and *false* at the level of individual tires. Extend a Heawood face-labelling to a connected tire cluster  $K$  by labelling every annular face of every tire of  $K$ , and for  $v \in V(K)$  write

$$\lambda_K^*(v) := \sum_f \lambda(f) \pmod{3},$$

the sum over the annular faces of  $K$  incident to  $v$ .

**Proposition 3.6** (Two-sided cluster decomposition at a vertex). *Let  $v \in V(G)$  have level  $\ell = \ell_G(v)$ , and let  $K_\ell$  and  $K_{\ell-1}$  be the at most two connected tire clusters containing  $v$ , of depths  $\ell$  and  $\ell-1$  respectively (Proposition 2.4). Then the bounded faces of  $G$  incident to  $v$  partition into the annular faces of  $K_\ell$  at  $v$  and the annular faces of  $K_{\ell-1}$  at  $v$ , and*

$$\sum_{f \ni v} \lambda(f) \equiv \lambda_{K_\ell}^*(v) + \lambda_{K_{\ell-1}}^*(v) \pmod{3}.$$

Each one-sided value  $\lambda_{K_d}^*(v)$  is the complete sum over all depth- $d$  faces at  $v$ , so the Heawood condition (3.1) at  $v$  reads

$$\lambda_{K_\ell}^*(v) + \lambda_{K_{\ell-1}}^*(v) \equiv 0 \pmod{3},$$

a pairing between the single child cluster  $K_\ell$  and the single parent cluster  $K_{\ell-1}$ . (When  $\ell = 0$ , or when  $v$  bounds no depth- $\ell$  face, only one term is present.)

*Proof.* By Proposition 2.4 (Step 1) every bounded face incident to  $v$  has depth  $\ell - 1$  or  $\ell$ , partitioning the incident faces by depth; by Step 2 all depth- $\ell$  faces at  $v$  lie in the single cluster  $K_\ell$  and all depth- $(\ell - 1)$  faces at  $v$  in  $K_{\ell-1}$ . Hence the depth- $\ell$  part is exactly the annular faces of  $K_\ell$  at  $v$ , the depth- $(\ell - 1)$  part those of  $K_{\ell-1}$ , and summing  $\lambda$  over the two parts gives the identity; (3.1) is its vanishing.  $\square$

*Remark 3.7* (Failure at the tire level). Proposition 3.6 is what makes the binary parent/child pairing possible, and it requires the cluster. A vertex  $v$  may lie on many depth- $\ell$  tires — the unbounded case of Section 2 — and the per-tire value  $\lambda^*(v)$  of Definition 3.1 then records only the faces of *one* tire at  $v$ , a fragment of  $v$ 's face-star. No single child tire carries the complete depth- $\ell$  sum, so the label  $\sum_{f \ni v} \lambda(f)$  cannot be written as one child value plus one parent value, and per-tire compatibility (Definition 3.4) fails to assemble to (3.1). Clustering repairs this: Proposition 2.4 guarantees exactly one cluster meets  $v$  on each side, so  $\lambda_{K_\ell}^*(v)$  is the complete child contribution and  $\lambda_{K_{\ell-1}}^*(v)$  the complete parent contribution. Every vertex label is then realised as the combination of a single child-cluster value with a single parent-cluster value, and the pigeonhole programme below chains *nested connected tire clusters* rather than individual tires.

We write  $R_K$  for the *cluster Heawood restriction relation*: the set of (outer, inner) boundary Heawood sequence pairs realisable by a face-labelling of  $K$ , defined as in Definition 3.3 but with the outer and inner boundaries of the cluster and the complete one-sided values  $\lambda_K^*$  in place of a single tire's, read up to rotation and global sign-flip. By Proposition 3.6 two nested clusters are compatible along their shared interface exactly when the inner sequence of the parent is the pointwise negation mod 3 of the outer sequence of the child (after the orientation reversal of Definition 3.4).

**Conjecture 3.8** (Heawood chain-pigeonhole principle). *There is a function  $N(k)$  such that the following holds. Let*

$$K_0 \supset K_1 \supset \cdots \supset K_{N(k)}$$

*be a nested chain of connected tire clusters in  $\mathcal{T}(G, S)$  whose shared interfaces have length at most  $k$ . Then two adjacent cluster restriction relations  $R_{K_i}, R_{K_{i+1}}$  in the chain admit compatible face-labellings along their shared interface, after rotation and global sign-flip. Equivalently, the chain contains a local gluing step that cannot be obstructed by disjoint Heawood boundary restrictions.*

**Conjecture 3.9** (Heawood cluster route to the Four Colour Theorem). *For every plane triangulation  $G$  and every level source  $S$ , the cluster Heawood restriction relations  $\{R_K : K \text{ a connected tire cluster}\}$  admit a selection of face-labellings that is compatible along every cluster interface. By Proposition 3.6 and Remark 3.5 this yields a  $\{+1, -1\}$  face-labelling of  $G$  satisfying (3.1), hence  $G$  is properly 4-vertex-colourable.*

## 4. THE CONSTRAINT FLOOR

A nested substructure constrains its outer interface through the set of Heawood boundary sequences it can realise. By the self-similarity of the tire decomposition ([3]), the region  $G_T$  enclosed by a tire's outer cycle, away from the source, is itself a triangulated disk; we record how tightly any such disk can constrain its boundary. The bound below depends only on the disk triangulation, not on a tire-tree labelling.

**Definition 4.1** (Achievable boundary set of a disk). Let  $D$  be a triangulated disk whose boundary is a simple  $n$ -cycle  $C = (v_0, \dots, v_{n-1})$ . Call a Heawood face-labelling  $\lambda : F(D) \rightarrow \{+1, -1\}$  *interior-valid* if  $\sum_{f \ni w} \lambda(f) \equiv 0 \pmod{3}$  at every interior vertex  $w$  of  $D$  (no condition on  $C$ ). The *achievable boundary set* of  $D$  is

$$\Phi(D) := \{ (\lambda^*(v_0), \dots, \lambda^*(v_{n-1})) : \lambda \text{ interior-valid} \} \subseteq \{0, 1, -1\}^n.$$

**Proposition 4.2** (Constraint floor). *For every triangulated disk  $D$  with boundary an  $n$ -cycle,*

$$|\Phi(D)| \geq 2^{n-2},$$

*and the bound is attained — already by the triangulation of the  $n$ -gon with no interior vertices. Consequently no nested structure constrains the outer cycle below  $2^{n-2}$  achievable Heawood sequences; the trivial tire is already maximally constraining.*

*Proof of attainment.* Triangulate the  $n$ -gon as a fan from  $v_0$ , with faces  $\{v_0, v_i, v_{i+1}\}$  for  $1 \leq i \leq n-2$  and labels  $\lambda_i := \lambda(\{v_0, v_i, v_{i+1}\})$ ; there are no interior vertices, so every labelling is interior-valid. The induced boundary values are

$$\lambda^*(v_1) = \lambda_1, \quad \lambda^*(v_i) = \lambda_{i-1} + \lambda_i \quad (1 < i < n-1), \quad \lambda^*(v_{n-1}) = \lambda_{n-2}, \quad \lambda^*(v_0) = \sum_j \lambda_j.$$

From  $\lambda^*(v_1)$  and the relations  $\lambda_i = \lambda^*(v_i) - \lambda_{i-1}$  the tuple  $(\lambda_1, \dots, \lambda_{n-2}) \in \{+1, -1\}^{n-2}$  is recovered from the boundary sequence, so the map  $\lambda \mapsto \lambda^*|_C$  is injective and  $|\Phi(D)| = 2^{n-2}$ .  $\square$

*Remark 4.3.* Two consequences. First,  $\Phi(D)$  is a  $\mathbb{Z}/3$  zonotope — a projected cube, sign-closed but not a  $\text{GF}(3)$  subspace — and at the floor it has size  $2^{n-2}$  with affine hull of dimension  $n-2$ . Second, since the floor is exponential in the interface length  $n$ , a maximally-constraining child still offers  $2^{n-2}$  outer options, so the gluing of Conjecture 3.8 has the least slack at *short* interfaces (e.g.  $n=4$  leaves 4 options) and is easy at long ones; the difficulty of the programme is concentrated at short level cycles.

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