HEAWOOD RESTRICTIONS ON NESTED TIRE GRAPH DUALS

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Abstract.

1. Introduction

A classical theorem of Tait recasts the Four Colour Theorem in dual, edge-colouring terms: a plane triangulation G is properly 4-vertex-colourable if and only if its dual cubic graph G' is properly 3-edge-colourable. Thus a minimal counterexample to the Four Colour Theorem – a smallest triangulation admitting no proper 4-colouring – corresponds to a smallest cubic plane graph admitting no proper 3-edge-colouring.

This paper continues the series studying that structure through the lens of *nested level duals*. The foundational vocabulary — level sources, levels, the inner planar dual G' and its dual depth, and tire graphs — is developed in the companion paper [3]; we refer to that paper for those definitions and rely on them throughout. In particular we use, without restating, the notions of:

- *level source* S and G-vertex levels $\ell_G(v)$;
- the inner planar dual G' ([3, Definition 1.3]);
- *dual depth* $\delta_G(d_f)$ ([3, Definition 1.4]);
- *tire graph* $T = (B_{\text{out}}, O, E_{\text{ann}})$ with outer/inner boundaries and annular edges ([3, Definition 1.5]);
- the *tire-component lemma* ([3, Lemma 1.8]); and
- the *tire-tread partition theorem* ([3, Theorem 1.9]).

Throughout, $G = (V, E)$ is a plane maximal planar graph (a triangulation) with a fixed planar embedding Π_G. We write $|V| = n$, so $|E| = 3n - 6$ and G has $2n - 4$ triangular faces.

The classical input is Heawood's face-sum identity [1]: for any proper 3-edge-colouring of a cubic plane graph H, assigning each face of H a number in $\{+1, -1\}$ can be done so that the labels around every vertex of H sum to 0 (mod 3). In the triangulation G dual to H this becomes a $\{+1, -1\}$ labelling of the *faces* of G whose incident-face sum at every vertex of G vanishes mod 3. Our aim is to record what this restriction forces along the boundary cycles of a nested tire graph, and to formulate a chain-pigeonhole programme in this Heawood labelling parallel to the medial programme of [4].

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2. Connected tire clusters

The tire treads at a fixed depth partition the depth-d faces of G [3], but distinct depth-d tires need not be vertex-disjoint: a single vertex of G may lie on the source-side boundary of several depth-d tires at once (this occurs exactly when the depth-d faces around that vertex are split into more than one arc by depth-$(d-1)$ faces). We organise the depth-d tires by this sharing.

Lemma 2.1 (Same-depth tires meet only in vertices)**.** *Let $T \neq T'$ be two distinct tire treads at the same depth d in $\mathcal{T}(G,S)$, arising from connected components C', C'' of the depth-d dual subgraph G'_d. Then T and T' share no edge of G; any intersection $V(T) \cap V(T')$ consists of isolated vertices.*

Proof. An edge e of G shared by two depth-d annular faces f_1, f_2 is, by definition of the inner dual [3, Definition 1.3], a dual edge of G' joining d_{f_1} and d_{f_2}; since $\delta(d_{f_1}) = \delta(d_{f_2}) = d$, this edge lies in G'_d, so d_{f_1} and d_{f_2} belong to the same component of G'_d. Hence no edge of G is shared by annular faces of two *different* components, and distinct depth-d tires share no edge. Their intersection is therefore a set of isolated vertices. □

Definition 2.2 (Connected tire cluster)**.** Fix a nested tire decomposition $\mathcal{T}(G,S)$ and a depth d. On the set of depth-d tire treads define the relation

$$T \sim T' \quad \Longleftrightarrow \quad V(T) \cap V(T') \neq \varnothing.$$

A *connected tire cluster* at depth d is the subgraph of G

$$\mathsf{K} \;=\; \bigcup_i T_i \;\subseteq\; G$$

obtained as the union (of underlying plane graphs) of the tires in a single connected component $\{T_i\}$ of the transitive closure of $\sim$. A cluster consisting of a single tire is *trivial*; the connected tire clusters at depth d partition the depth-d tires.

Remark 2.3. By Lemma 2.1 the constituent tires of a connected tire cluster are joined only at shared vertices, each of which is a cut vertex of K; a connected tire cluster is thus a "cactus of tires" and is in general *not* itself a tire graph, since the annulus structure of [3, Definition 1.5] fails at each such pinch. The shared (cut) vertices are precisely the vertices that belong to more than one depth-d tire.

A single vertex may belong to several tires at one depth — the high-degree case where its depth-d faces split into many arcs — so the number of *tires* through a vertex is unbounded. Clustering collapses exactly this multiplicity: all tires through a vertex at a fixed depth share that vertex, hence lie in one cluster. The cluster count is therefore controlled.

Proposition 2.4 (A vertex meets at most two clusters)**.** *Every vertex $v \in V(G)$ belongs to at most two connected tire clusters, namely at most one at each of the two consecutive depths $\ell_G(v) - 1$ and $\ell_G(v)$. In particular a source vertex ($\ell_G(v) = 0$) belongs to a single cluster.*

Proof. Write $\ell = \ell_G(v)$.

Step 1: every bounded face incident to v has dual depth $\ell - 1$ or ℓ. Let f be a bounded triangular face with $v \in V(f)$. Then $\delta_G(d_f) = \min_{u \in V(f)} \ell_G(u) \leq \ell_G(v) = \ell$. The other two vertices of f are adjacent to v in G, and the level function

$\ell_G(\cdot) = \mathrm{dist}_G(\cdot, S)$ is 1-Lipschitz along edges, so each has level at least $\ell - 1$; hence $\delta_G(d_f) \geq \ell - 1$. Thus $\delta_G(d_f) \in \{\ell - 1, \ell\}$ (only $\delta_G(d_f) = 0$ when $\ell = 0$), so v bounds faces of, and therefore belongs to tires of, no depth other than $\ell - 1$ or ℓ.

Step 2: at each depth, all tires through v lie in one cluster. Fix $d \in \{\ell - 1, \ell\}$ and let T, T' be depth-d tires with $v \in V(T) \cap V(T')$. Then $V(T) \cap V(T') \neq \varnothing$, so $T \sim T'$ in the sense of Definition 2.2, and all depth-d tires containing v lie in a single connected component of $\sim$ — one connected tire cluster K_d.

Combining the two steps, v belongs to at most the clusters $\mathsf{K}_{\ell-1}$ and K_ℓ, i.e. to at most two connected tire clusters; when $\ell = 0$ only K_0 occurs. □

3. Heawood restrictions on the tire dual

We work inside a fixed nested tire decomposition $\mathcal{T}(G, S)$ of G from a single-vertex level source S [3], and use the tire data $T = (B_{\mathrm{out}}, O, E_{\mathrm{ann}})$ with annular faces F_{ann}, outer boundary B_{out}, and inner boundary B_{in} ([3, Definition 1.5]). Since O is outerplanar, every vertex of a tire lies on B_{out} or on the inner-boundary walk B_{in}; a tire has no interior vertices.

Definition 3.1 (Heawood face-labelling of a tire)**.** A *Heawood face-labelling* of a tire graph T is a map

$$\lambda : F_{\mathrm{ann}} \longrightarrow \{+1, -1\}$$

assigning a sign to each annular face of T. For a vertex $v \in V(T)$, write $F_{\mathrm{ann}}(v) \subseteq F_{\mathrm{ann}}$ for the set of annular faces of T incident to v, and define the *induced vertex value*

$$\lambda^*(v) \;:=\; \sum_{f \in F_{\mathrm{ann}}(v)} \lambda(f) \mod 3 \;\in\; \{0, 1, -1\}.$$

The value $\lambda^*(v)$ is the *partial* face-sum at v taken over the annular faces of T alone, not over all faces of G incident to v.

Remark 3.2. Because a tire has no interior vertices, every annular face of T is incident to $B_{\mathrm{out}} \cup B_{\mathrm{in}}$, and a Heawood face-labelling is subject to *no* internal constraint: all $2^{|F_{\mathrm{ann}}|}$ sign assignments are admissible. The Heawood restriction is felt only on the two boundary cycles, through the induced vertex values λ^*.

Definition 3.3 (Induced boundary sequences)**.** Let λ be a Heawood face-labelling of T. Reading the vertices of B_{out} in clockwise order $v_0, v_1, \ldots, v_{p-1}$, the *outer Heawood sequence* of (T, λ) is

$$\sigma_{\mathrm{out}}(T, \lambda) \;:=\; \big(\lambda^*(v_0), \ldots, \lambda^*(v_{p-1})\big) \;\in\; \{0, 1, -1\}^p.$$

Reading the inner-boundary walk B_{in} in clockwise order $w_0, \ldots, w_{q-1}$ gives the *inner Heawood sequence* $\sigma_{\mathrm{in}}(T, \lambda) \in \{0, 1, -1\}^q$. The *Heawood restriction relation* of T is the set

$$R_T \;:=\; \big\{ \big(\sigma_{\mathrm{out}}(T, \lambda),\, \sigma_{\mathrm{in}}(T, \lambda)\big) \;:\; \lambda : F_{\mathrm{ann}} \to \{+1, -1\} \big\}$$

of all (outer, inner) sequence pairs realisable by a single face-labelling, read up to rotation and the global sign-flip $\lambda \mapsto -\lambda$ (equivalently $\sigma \mapsto -\sigma$).

Definition 3.4 (Heawood compatibility across an interface)**.** Let T be a tire and $T' \in \mathcal{T}(G, S)$ a child of T, so the outer boundary cycle $B_{\mathrm{out}}^{(T')}$ coincides with a bounded face of $O^{(T)}$; let γ be this shared cycle, of length L, and let v range over its vertices. Heawood face-labellings λ of T and λ' of T' are *compatible along* γ if at every shared vertex v,

$$\lambda^*(v) + (\lambda')^*(v) \;\equiv\; 0 \pmod 3,$$

i.e. 0 is paired with 0 and +1 with −1. Equivalently, the inner Heawood sequence of T on γ is the pointwise negation mod 3 of the outer Heawood sequence of T' on γ, after reversing one of the two clockwise readings to account for the opposite rotational senses in which T and T' traverse γ.

Remark 3.5. Call v *interior* if it is not incident to the outer face of Π_G. For an interior vertex every incident face is bounded, and compatibility along γ at v is exactly the statement that the incident-face sum at v — over the parent's annular faces together with the child's — vanishes mod 3:

$$\sum_{f \ni v} \lambda(f) \;\equiv\; 0 \pmod 3 \qquad \text{for every interior vertex } v \in V(G), \tag{3.1}$$

the sum ranging over the bounded faces incident to v. The interfaces of $\mathcal{T}(G, S)$ are interior level cycles, so cluster compatibility only ever constrains interior vertices and is untouched by the outer face.

To pass from (3.1) to a colouring one must account for the outer face: an outer-boundary vertex is incident to the unbounded face f_∞, whose label is omitted from the bounded sum. Extend λ by a single label $\lambda(f_\infty) \in \{+1, -1\}$ on f_∞. Then a family of Heawood face-labellings that is pairwise compatible along every interface of $\mathcal{T}(G, S)$ assembles into a $\{+1, -1\}$ labelling of *all* faces of G for which $\sum_{f \ni v} \lambda(f) \equiv 0 \pmod 3$ holds at every vertex — the outer-boundary vertices now carrying $\lambda(f_\infty)$ in their sum. This is Heawood's face-sum identity [1] for a proper 3-edge-colouring of the full cubic dual of G, hence (by Tait) a proper 4-vertex-colouring of G.

Why the programme runs between nested clusters. The vanishing condition (3.1) at a vertex v is a constraint on the *full* face-star of v. To run a pigeonhole between two objects — a child and a parent — we need that full sum to split as exactly two one-sided contributions, so that each vertex label is the combination of a single child value and a single parent value. This is true at the level of connected tire clusters, and *false* at the level of individual tires. Extend a Heawood face-labelling to a connected tire cluster K by labelling every annular face of every tire of K, and for $v \in V(\mathsf{K})$ write

$$\lambda^*_{\mathsf{K}}(v) \;:=\; \sum_{f} \lambda(f) \mod 3,$$

the sum over the annular faces of K incident to v.

Proposition 3.6 (Two-sided cluster decomposition at a vertex)**.** *Let $v \in V(G)$ have level $\ell = \ell_G(v)$, and let K_ℓ and $\mathsf{K}_{\ell-1}$ be the at most two connected tire clusters containing v, of depths ℓ and $\ell - 1$ respectively (Proposition 2.4). Then the bounded faces of G incident to v partition into the annular faces of K_ℓ at v and the annular faces of $\mathsf{K}_{\ell-1}$ at v, and*

$$\sum_{f \ni v} \lambda(f) \;\equiv\; \lambda^*_{\mathsf{K}_\ell}(v) + \lambda^*_{\mathsf{K}_{\ell-1}}(v) \pmod 3.$$

Each one-sided value $\lambda^*_{\mathsf{K}_d}(v)$ is the complete *sum over all depth-d faces at v, so the Heawood condition* (3.1) *at v reads*

$$\lambda^*_{\mathsf{K}_\ell}(v) + \lambda^*_{\mathsf{K}_{\ell-1}}(v) \equiv 0 \pmod 3,$$

a pairing between the single child cluster K_ℓ and the single parent cluster $\mathsf{K}_{\ell-1}$. (When $\ell = 0$, or when v bounds no depth-ℓ face, only one term is present.)

Proof. By Proposition 2.4 (Step 1) every bounded face incident to v has depth $\ell-1$ or ℓ, partitioning the incident faces by depth; by Step 2 all depth-ℓ faces at v lie in the single cluster K_ℓ and all depth-$(\ell-1)$ faces at v in $\mathsf{K}_{\ell-1}$. Hence the depth-ℓ part is exactly the annular faces of K_ℓ at v, the depth-$(\ell-1)$ part those of $\mathsf{K}_{\ell-1}$, and summing λ over the two parts gives the identity; (3.1) is its vanishing. □

Remark 3.7 (Failure at the tire level). Proposition 3.6 is what makes the binary parent/child pairing possible, and it requires the cluster. A vertex v may lie on many depth-ℓ tires — the unbounded case of Section 2 — and the per-tire value $\lambda^*(v)$ of Definition 3.1 then records only the faces of *one* tire at v, a fragment of v's face-star. No single child tire carries the complete depth-ℓ sum, so the label $\sum_{f\ni v}\lambda(f)$ cannot be written as one child value plus one parent value, and per-tire compatibility (Definition 3.4) fails to assemble to (3.1). Clustering repairs this: Proposition 2.4 guarantees exactly one cluster meets v on each side, so $\lambda^*_{\mathsf{K}_\ell}(v)$ is the complete child contribution and $\lambda^*_{\mathsf{K}_{\ell-1}}(v)$ the complete parent contribution. Every vertex label is then realised as the combination of a single child-cluster value with a single parent-cluster value, and the pigeonhole programme below chains *nested connected tire clusters* rather than individual tires.

We write R_{K} for the *cluster Heawood restriction relation*: the set of (outer, inner) boundary Heawood sequence pairs realisable by a face-labelling of K, defined as in Definition 3.3 but with the outer and inner boundaries of the cluster and the complete one-sided values λ^*_{K} in place of a single tire's, read up to rotation and global sign-flip. By Proposition 3.6 two nested clusters are compatible along their shared interface exactly when the inner sequence of the parent is the pointwise negation mod 3 of the outer sequence of the child (after the orientation reversal of Definition 3.4).

Conjecture 3.8 (Heawood chain-pigeonhole principle)**.** *There is a function $N(k)$ such that the following holds. Let*

$$\mathsf{K}_0 \supset \mathsf{K}_1 \supset \cdots \supset \mathsf{K}_{N(k)}$$

be a nested chain of connected tire clusters in $\mathcal{T}(G, S)$ whose shared interfaces have length at most k. Then two adjacent cluster restriction relations $R_{\mathsf{K}_i}, R_{\mathsf{K}_{i+1}}$ in the chain admit compatible face-labellings along their shared interface, after rotation and global sign-flip. Equivalently, the chain contains a local gluing step that cannot be obstructed by disjoint Heawood boundary restrictions.

Conjecture 3.9 (Heawood cluster route to the Four Colour Theorem)**.** *For every plane triangulation G and every level source S, the cluster Heawood restriction relations $\{R_{\mathsf{K}} : \mathsf{K}$ a connected tire cluster$\}$ admit a selection of face-labellings that is compatible along every cluster interface. By Proposition 3.6 and Remark 3.5 this yields a $\{+1, -1\}$ face-labelling of G satisfying* (3.1)*, hence G is properly* 4*-vertex-colourable.*

4. The constraint floor

A nested substructure constrains its outer interface through the set of Heawood boundary sequences it can realise. By the self-similarity of the tire decomposition ([3]), the region G_T enclosed by a tire's outer cycle, away from the source, is itself a triangulated disk; we record how tightly any such disk can constrain its boundary. The bound below depends only on the disk triangulation, not on a tire-tree labelling.

Definition 4.1 (Achievable boundary set of a disk)**.** Let D be a triangulated disk whose boundary is a simple n-cycle $C = (v_0, \ldots, v_{n-1})$. Call a Heawood face-labelling $\lambda : F(D) \to \{+1, -1\}$ *interior-valid* if $\sum_{f \ni w} \lambda(f) \equiv 0 \pmod 3$ at every interior vertex w of D (no condition on C). The *achievable boundary set* of D is

$$\Phi(D) \;:=\; \big\{\, (\lambda^*(v_0), \ldots, \lambda^*(v_{n-1})) \;:\; \lambda \text{ interior-valid} \,\big\} \;\subseteq\; \{0, 1, -1\}^n.$$

Proposition 4.2 (Constraint floor)**.** *For every triangulated disk D with boundary an n-cycle,*

$$|\Phi(D)| \;\geq\; 2^{\,n-2},$$

and the bound is attained — already by the triangulation of the n-gon with no interior vertices. Consequently no nested structure constrains the outer cycle below $2^{\,n-2}$ achievable Heawood sequences; the trivial tire is already maximally constraining.

Proof of attainment. Triangulate the n-gon as a fan from v_0, with faces $\{v_0, v_i, v_{i+1}\}$ for $1 \leq i \leq n-2$ and labels $\lambda_i := \lambda(\{v_0, v_i, v_{i+1}\})$; there are no interior vertices, so every labelling is interior-valid. The induced boundary values are

$$\lambda^*(v_1) = \lambda_1, \quad \lambda^*(v_i) = \lambda_{i-1} + \lambda_i \;\; (1 < i < n-1), \quad \lambda^*(v_{n-1}) = \lambda_{n-2}, \quad \lambda^*(v_0) = \textstyle\sum_j \lambda_j.$$

From $\lambda^*(v_1)$ and the relations $\lambda_i = \lambda^*(v_i) - \lambda_{i-1}$ the tuple $(\lambda_1, \ldots, \lambda_{n-2}) \in \{+1, -1\}^{n-2}$ is recovered from the boundary sequence, so the map $\lambda \mapsto \lambda^*|_C$ is injective and $|\Phi(D)| = 2^{\,n-2}$. ∎

Remark 4.3 (Depth is freedom-positive). The lower bound is plausible from a counting balance. A triangulated disk with k interior vertices has $2k+n-2$ faces (Euler) and imposes exactly k interior Heawood constraints, one per interior vertex. So each interior vertex contributes *two* faces — two new $\{+1, -1\}$ degrees of freedom — against only *one* constraint, and the free dimension $(2k+n-2)-k = k+n-2$ *grows* with depth. Going deeper is freedom-positive on balance: the boundary projection $\Phi(D)$ can only retain or enlarge its options, never drop below the interior-free value $2^{\,n-2}$. (Empirically $|\Phi(D)|$ does grow with k; e.g. on the 4-cycle the central-apex wheel realises 5 sequences against the fan's 4.) The constraints relate only interior-incident faces and cannot collapse the $n-2$ degrees of freedom carried by the boundary-incident faces — which is the content the lower bound must make precise.

Remark 4.4. Two consequences. First, $\Phi(D)$ is a $\mathbb{Z}/3$ zonotope — a projected cube, sign-closed but not a GF(3) subspace — and at the floor it has size $2^{\,n-2}$ with affine hull of dimension $n-2$. Second, since the floor is exponential in the interface length n, a maximally-constraining child still offers $2^{\,n-2}$ outer options, so the gluing of Conjecture 3.8 has the least slack at *short* interfaces (e.g. $n = 4$ leaves 4 options) and is easy at long ones; the difficulty of the programme is concentrated at short level cycles.

References

[1] P. J. Heawood, *On the four-colour map theorem*, Quart. J. Pure Appl. Math. **29** (1898), 270–285.
[2] E. Bauerfeld, *Plane Depth*, manuscript (math-research repository), 2026.
[3] E. Bauerfeld, *Nested Tire Decompositions of Plane Triangulations*, manuscript (math-research repository), 2026.
[4] E. Bauerfeld, *Medial Tire Decompositions of Plane Triangulations*, manuscript (math-research repository), 2026.
[5] E. Bauerfeld, *Coloring Nested Tire Dual Graphs*, manuscript (math-research repository), 2026.