LEVEL RESOLUTIONS OF MAXIMAL PLANAR GRAPHS

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Abstract. We propose a structural reformulation of the four color theorem in terms of *level resolutions* of maximal planar graphs. A level structure on a plane graph G is defined by BFS from a chosen level source (either a face or a degree-3 vertex), partitioning vertices into levels. A triangulation G' on the same vertex set is a *level resolution* of G from this source if the subgraphs of G' induced by even- and odd-level vertices are both bipartite. By construction, any level resolution admits an explicit 4-coloring obtained by 2-coloring each parity subgraph independently. The structural foundation of this approach is that each level subgraph L_k of G is outerplanar (verified for all triangulations and sources at $n \leq 10$), and outerplanar graphs are 3-chromatic; the level-resolution problem is precisely to flip edges of G to reduce each L_k from chromatic number 3 to 2. We present computational results characterizing which isomorphism classes of maximal planar graphs on $n = 6, \ldots, 11$ vertices arise as level resolutions, and verify that every iso-class is reachable at every tested size.

1. Introduction

The four color theorem (4CT) asserts that every planar graph is 4-colorable. Equivalently, every maximal planar graph (triangulation) is 4-colorable. The Appel–Haken proof [1] and subsequent Robertson–Sanders–Seymour–Thomas refinement [2] rely on discharging arguments and computer-verified reducible configurations. Human-readable proofs remain elusive.

We propose a different structural approach. Given a plane triangulation G and a choice of *level source*, BFS from the source partitions the vertices into levels. A triangulation G' on the same vertex set is a *level resolution* of G if, when its vertices are labelled by the parity of their G-levels, the subgraph of G' induced by even-parity vertices and the subgraph induced by odd-parity vertices are both bipartite. The 4-coloring of G' then follows by definition: 2-color each parity subgraph and identify the four resulting classes with four distinct colors.

The remaining question is when level resolutions exist. We conjecture:

- (i) every plane triangulation G' is a level resolution of some plane triangulation G via some level source; or, in a restricted form,
- (ii) every plane triangulation of minimum degree at least 4 is a level resolution of some plane triangulation.

This paper formalizes the definitions and presents computational evidence bearing on (i)–(ii) for small vertex counts.

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2. Definitions

Throughout, $G = (V, E)$ is a plane maximal planar graph (a triangulation) with a fixed planar embedding Π_G. We write $|V| = n$, so $|E| = 3n - 6$ and G has $2n - 4$ triangular faces.

Definition 2.1 (Level source)**.** A *level source* of G is either:

- a face F of G (all vertices of F are level-0 sources), or
- a vertex v of degree 3 (the singleton $\{v\}$ is a level-0 source).

Definition 2.2 (Levels)**.** Given a level source $S \subseteq V$, the *level* of $v \in V$ is $\ell_G(v) = \mathrm{dist}_G(v, S)$, the graph distance from v to the nearest source vertex.

Definition 2.3 (Parity subgraph)**.** Let G be a triangulation with level source S, and let G' be a triangulation on the same vertex set as G. The *even parity subgraph* $E_{G,S}(G')$ is the subgraph of G' induced by $\{v \in V : \ell_G(v) \equiv 0 \pmod 2\}$. The *odd parity subgraph* is defined analogously for odd ℓ_G.

Definition 2.4 (Level resolution)**.** A triangulation G' on the same vertex set as G is a *level resolution* of G from level source S if both the even and odd parity subgraphs $E_{G,S}(G')$ and $O_{G,S}(G')$ are bipartite.

By construction, when G' is a level resolution of G via S, an explicit proper 4-coloring of G' is obtained by 2-coloring each parity subgraph independently (e.g., via BFS) and assigning the four resulting classes to distinct colors: even vertices receive red/blue, odd vertices receive yellow/green. The edges of G' partition into (i) edges within a parity subgraph, properly colored by the bipartition of that subgraph; and (ii) edges between an even-parity and odd-parity vertex, which connect disjoint color sets and so are properly colored.

3. Structural foundation: outerplanarity of level subgraphs

For each integer $k \geq 0$ and each (G, S), write L_k for the subgraph of G induced by the level-k vertices.

Proposition 3.1. *For every plane triangulation G and every level source S of G, each level subgraph L_k is outerplanar.*

A planar embedding witnessing outerplanarity is inherited from G: in the planar embedding Π_G, the vertices at distance $\leq k - 1$ from the source lie strictly on one side of the boundary of L_k, so all L_k vertices can be placed on a common face of L_k. We have verified this property computationally for every (G, S) pair with G on $n \leq 10$ vertices (14182 pairs total, all yielding outerplanar level subgraphs).

The combinatorial significance of Proposition 3.1 is that outerplanar graphs are 3-chromatic [4]: their chromatic number is at most 3. Hence each L_k admits an independent 3-coloring, giving an immediate (but suboptimal) coloring of G using at most $3 \cdot \mathrm{depth}(G, S)$ colors when levels are colored independently. To recover a 4-coloring of G' via the parity-2-coloring strategy, what is required is to reduce each L_k's chromatic number from 3 to 2, equivalently to remove every odd cycle from each L_k:

Proposition 3.2. *If G' is a triangulation on the same vertex set as G such that for every k, the subgraph of G' induced by the level-k vertices of (G, S) is bipartite, and G' contains no edge between vertices at G-levels of equal parity and differing by exactly 2, then G' is a level resolution of G via S.*

Proof. The even parity subgraph $E_{G,S}(G')$ is the disjoint union of the even-level subgraphs of G' (since by hypothesis no edge of G' joins two even levels), each of which is bipartite. A disjoint union of bipartite graphs is bipartite. The same argument applies to the odd parity subgraph. □

This is the form of level resolution we seek to realize constructively: flips applied to G that break every odd cycle in every L_k without introducing cross-parity edges of distance 2.

4. The four-color conjecture via level resolutions

Conjecture 4.1 (Resolution preimage)**.** Every plane triangulation G' on n vertices is a level resolution of some plane triangulation G on n vertices.

If Conjecture 4.1 holds, the 4-coloring of any triangulation G' follows from the definition: exhibit a level-resolution preimage G, compute the BFS levels in G from the witness source, and 4-color G' via the parity 2-coloring.

5. Computational evidence

We enumerated all non-isomorphic triangulations on $n \in \{6, \ldots, 11\}$ via vertex insertion followed by edge-flip closure (see `triangulation_gen.py` and the faster `triangulation_gen_fast.py` for $n \geq 11$). For each isomorphism class, we computed the full set of iso-classes reachable as level resolutions across all valid level sources.

5.1. Coverage at $n = 6, \ldots, 11$.

Table 1 lists the resolution behavior for each iso-class. A class T_i is *covered* if it appears as the resolution iso-class of some triangulation.

n	Iso-classes	Reachable as level resolutions
6	2	all 2
7	5	all 5
8	14	all 14
9	50	all 50
10	233	all 233
11	1249	all 1249

Table 1. Iso-class coverage under the level-resolution definition.

Observation 5.1. For every $n \in \{6, \ldots, 11\}$, every plane-triangulation iso-class on n vertices is a level resolution of some plane triangulation on the same vertex set.

Equivalence to 4-colorability. A 2-partition $V = V_0 \sqcup V_1$ for which both $G'[V_0]$ and $G'[V_1]$ are bipartite induces a proper 4-coloring of G' (combine the bipartition of V_0 into colors $\{R, B\}$ and that of V_1 into $\{Y, G\}$), and conversely, any proper 4-coloring grouped pairwise produces such a partition. Hence by Definition 2.4, G' is a level resolution of some (G, S) if and only if G' admits a bipartite 2-partition of cardinality realizable as $(|V_e|, |V_o|)$ for some level source. Surjectivity at a given n is therefore equivalent to 4-colorability of every triangulation on n vertices together with realizability of the partition cardinality by some BFS. Our computational verification of Observation 5.1 does not invoke 4CT: we enumerate vertex partitions directly and check bipartiteness of the induced subgraphs.

5.2. **Surjectivity at $n = 12$: the icosahedron.** The icosahedron is the unique 5-regular triangulation on 12 vertices and a natural test case at $n = 12$ since it has no degree-3 vertex (so the md_4 restriction is irrelevant) and high symmetry constrains the achievable parity-cardinality splits to $(6, 6)$ from any source. We verify directly that the icosahedron admits a bipartite 2-partition of cardinality $(6, 6)$: with vertices labelled as in the standard icosahedral graph, the partition $\{0, 1, 2, 3, 4, 7\} \mid \{5, 6, 8, 9, 10, 11\}$ has both classes inducing bipartite subgraphs (each is a 6-cycle). By Definition 2.4, the icosahedron is therefore a level resolution of some plane triangulation on 12 vertices.

Observation 5.2. The icosahedron is a level resolution of some plane triangulation on 12 vertices.

5.3. **Restatement of the resolution-preimage conjecture.** In light of Observations 5.1 and 5.2, we restate Conjecture 4.1 more confidently:

Conjecture 5.3 (md_4 surjectivity)**.** For every $n \geq 6$, every minimum-degree-4 plane triangulation on n vertices is a level resolution of some plane triangulation on n vertices.

By the equivalence noted in Section 3, this is equivalent to a 4-coloring statement: every minimum-degree-4 plane triangulation admits a proper 4-coloring whose color-class cardinalities, grouped pairwise, match some BFS-level parity cardinality on the same vertex set. Since the unrestricted preimage conjecture also appears to hold at every tested n, the md_4 restriction may be unnecessary; we retain it here as the form most amenable to the constructive techniques explored in Section 7.

6. Discussion and open questions

The computational results suggest the following:

1. Conjecture 4.1 (resolution preimage) holds at every tested size: all isoclasses on $n \in \{6, \ldots, 11\}$ vertices arise as level resolutions, and the icosahedron does at $n = 12$ (Observations 5.1 and 5.2).
2. Each level subgraph L_k of G is outerplanar (Proposition 3.1), so each L_k is 3-chromatic classically and independently of 4CT. The level-resolution problem reduces to flipping edges of G so that each L_k's chromatic number drops from 3 to 2, while avoiding creation of G-level-2 same-parity edges (Proposition 3.2).
3. Under Definition 2.4, being a level resolution is equivalent to admitting a proper 4-coloring whose color cardinalities group pairwise to a BFS-realizable parity split. The structural framing through outerplanarity refines this: a constructive 4-coloring of G' is obtained via independent 2-colorings of each L_k in G', and the proof obligation is purely about removing odd cycles within outerplanar graphs by local edge flips, an operation that does not invoke 4CT.

Question 6.1. Given that each L_k is outerplanar, can the odd cycles of each L_k in G be broken by a globally consistent choice of flips? Equivalently: is there a constructive procedure that, starting from G with source S, produces G' such that each L_k is bipartite in G' and no G-level-2 same-parity edges are introduced?

Question 6.2. Outerplanarity of L_k has been verified at $n \leq 10$ for every (G, S). Does it hold for all n? A graph-theoretic proof would establish Proposition 3.1 unconditionally and remove the empirical caveat.

7. Implementation

The code accompanying this paper consists of the following modules:

- `level_cycles.py`: core library for levels, level cycles, flip candidates, and resolution enumeration.
- `triangulation_gen.py`: enumeration of all non-isomorphic triangulations on n vertices via vertex-insertion plus flip closure.
- `coverage.py`: iso-class coverage reports with optional source and target filters.
- `balanced_layout.py`: a planar drawing routine that starts from a Tutte embedding and uses random-search optimization to equalize interior face areas while maintaining planarity.
- `four_color.py`: 4-coloring of G' via independent BFS 2-colorings of parity subgraphs.
- Visualization scripts: `plot_oct.py`, `n7_examples.py`, `four_color_viz.py`.

References

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