EVEN LEVEL GRAPH GENERATORS

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Abstract.

1. Introduction

2. Definitions

Throughout, $G = (V, E)$ is a plane maximal planar graph (a triangulation) with a fixed planar embedding Π_G. We write $|V| = n$, so $|E| = 3n - 6$ and G has $2n - 4$ triangular faces.

Definition 2.1 (Level source)**.** A *level source* of G is any vertex $v \in V$; we write $S = \{v\}$ for the level-0 source.

Definition 2.2 (Levels)**.** Given a level source $S \subseteq V$, the *level* of $v \in V$ is $\ell_G(v) = \mathrm{dist}_G(v, S)$, the graph distance from v to the nearest source vertex.

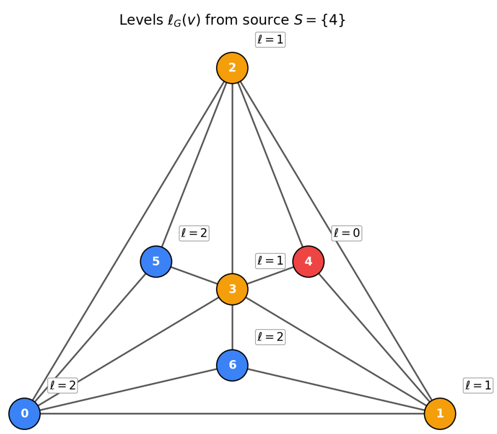

Figure 1. BFS levels from the degree-3 vertex source $S = \{4\}$. The source is level 0, its three neighbours are level 1, and the remaining vertices are level 2. Colour encodes the level.

Definition 2.3 (Level cycle)**.** A *level cycle* of G (with respect to a level source S) is a simple cycle in G all of whose vertices have the same level.

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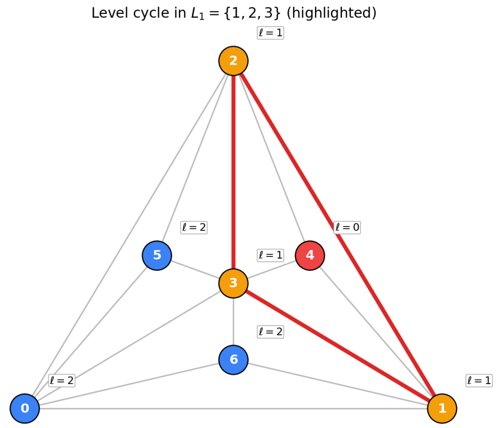

Figure 2. A level cycle in the triangulation of Figure 1. The triangle 1–2–3 is a simple cycle whose three vertices all lie at level 1, so it is a level cycle at level 1.

Definition 2.4 (Edge switch)**.** Let G be a triangulation with level source S, and let $e = uv$ be an edge of a level cycle of G. The *edge switch* at e is the edge flip on e: writing uvw and uvx for the two triangular faces of G containing e, the edge uv is removed and the edge wx is added. As with any edge flip, the result is a triangulation on the same vertex set provided w and x are non-adjacent in G.

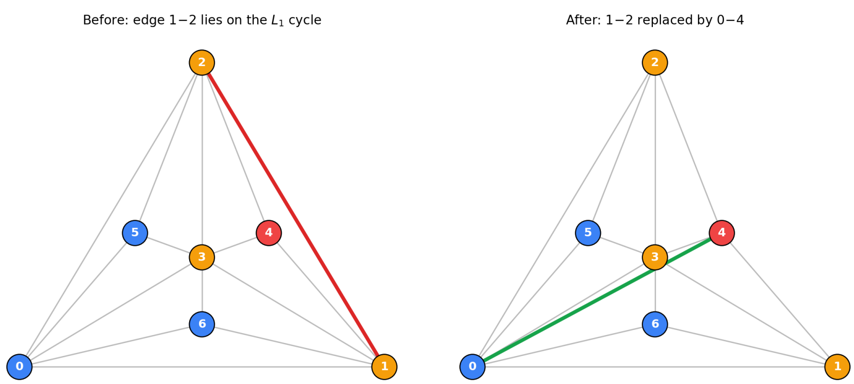

Figure 3. An edge switch on the level cycle of Figure 2. The chosen cycle edge $1-2$ is shared by the triangular faces $(0, 1, 2)$ and $(1, 2, 4)$; the switch deletes $1-2$ (red, left) and inserts $0-4$ (green, right). Vertex colours indicate the original levels in G.

Definition 2.5 (Parity subgraph)**.** Let G be a triangulation with level source S, and let G' be a triangulation on the same vertex set as G. The *even parity subgraph* $E_{G,S}(G')$ is the subgraph of G' induced by $\{v \in V : \ell_G(v) \equiv 0 \pmod 2\}$. The *odd parity subgraph* is defined analogously for odd ℓ_G.

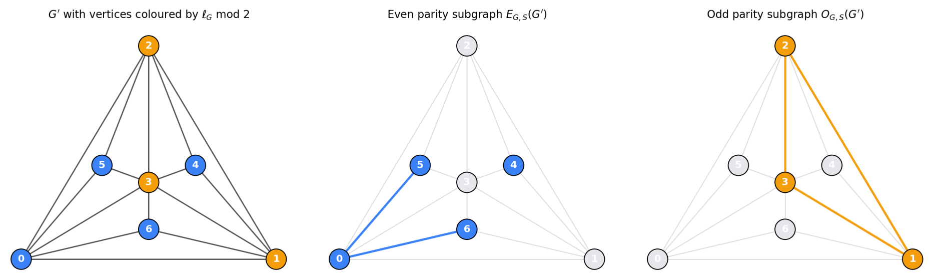

FIGURE 4. Parity subgraphs of $G' = T$ with respect to the level structure of Figure 1 (here we take $G = G' = T$). Left: T with vertices coloured by ℓ_G mod 2 (blue = even, orange = odd). Middle: the even parity subgraph $E_{G,S}(G')$, induced on $\{0, 4, 5, 6\}$; only edges with both endpoints even appear. Right: the odd parity subgraph $O_{G,S}(G')$, induced on $\{1, 2, 3\}$; the highlighted triangle shows that $O_{G,S}(G')$ is not bipartite for this choice of G'.

3. Outerplanarity of level components

For each integer $k \geq 0$ and each (G, S), write L_k for the subgraph of G induced by the level-k vertices. A *level component* of G (with respect to S) is a connected component of some L_k.

Theorem 3.1. *For every plane triangulation G and every level source S of G, every level component of G is outerplanar.*

Proof. Since every subgraph of an outerplanar graph is outerplanar, it suffices to show that each level subgraph L_k is outerplanar. For $k = 0$, $L_0 = S$ is a single vertex and is trivially outerplanar.

Fix $k \geq 1$ and let D_k be the drawing of L_k inherited from Π_G. Let F^* be the face of D_k containing the source. Suppose for contradiction that some $u \in L_k$ does not lie on ∂F^*, so u lies on the boundary of some other face of D_k. Take any path P in G from $v_0 \in S$ to u. As a curve in Π_G, P starts in F^* and ends at a point off ∂F^*, so it must transition from F^* to a different face of D_k; in a planar embedding this can happen only at a vertex of D_k, that is, at a level-k vertex w on P. Either $w \neq u$ (so P has length $\geq \text{dist}_G(S, w) + 1 \geq k + 1$), or $w = u$ (contradicting $u \notin \partial F^*$). Since every S-to-u path has length $\geq k + 1$, $\text{dist}_G(S, u) \geq k + 1$, contradicting $u \in L_k$. □

4. Even Level Graphs

Definition 4.1 (Even Level Graph)**.** A plane triangulation G with level source S is an *Even Level Graph* if every level cycle of G has even length.

Theorem 4.2. *Every Even Level Graph is* 4*-colorable.*

Proof. Since adjacent vertices in G have levels differing by at most 1, any edge between two same-parity endpoints in fact connects two vertices at the same level. Hence

$$E_{G,S}(G) = \bigsqcup_{i \geq 0} L_{2i}, \qquad O_{G,S}(G) = \bigsqcup_{i \geq 0} L_{2i+1},$$

and each L_k is bipartite because its cycles are level cycles of G, which have even length by hypothesis. Choose a 2-coloring of $E_{G,S}(G)$ in $\{\text{red}, \text{blue}\}$ and a 2-coloring of $O_{G,S}(G)$ in $\{\text{yellow}, \text{green}\}$. Same-parity edges of G are properly colored by the respective bipartition; opposite-parity edges connect $\{\text{red}, \text{blue}\}$ to $\{\text{yellow}, \text{green}\}$. The combined assignment is a proper 4-coloring of G. □

Definition 4.3 (Derived level graph)**.** Let G be an Even Level Graph with level source S, and let E and O denote the edge sets of the even and odd parity subgraphs $E_{G,S}(G)$ and $O_{G,S}(G)$. A *derived level graph* of G is a triangulation G' on the same vertex set as G obtained by a sequence of edge switches (Definition 2.4), each acting on an edge of E or of O. We do not update E or O to reflect the level structure of intermediate triangulations: throughout the sequence, an edge is classified as belonging to E (resp. O) if and only if both of its endpoints have even (resp. odd) level in G.

A derived level graph G' is *valid* if both $E_{G,S}(G')$ and $O_{G,S}(G')$ contain only even cycles.

Definition 4.4 (Bridge switch)**.** Let G' be a triangulation reached from an Even Level Graph G, with parity classes inherited from G as in Definition 4.3. An edge switch on an edge $e \in E \cup O$ of G', replacing uvw, uvx by the edge wx, is a *bridge switch* if either

- the new edge wx is a cross-parity edge (one endpoint even, the other odd), so wx enters neither parity subgraph; or
- wx is a same-parity edge and is a *bridge* in the parity subgraph it joins – that is, w and x lie in different connected components of that parity subgraph, so adding wx creates no new cycle.

Definition 4.5 (Bridge-derived level graph)**.** A *bridge-derived level graph* of an Even Level Graph G is a triangulation obtained from G by a sequence of bridge switches (Definition 4.4).

Because a bridge switch never closes a cycle in a parity subgraph, it never introduces an odd cycle there. As an Even Level Graph has bipartite parity subgraphs (every level cycle is even), every bridge-derived level graph has bipartite parity subgraphs as well, and so is automatically a valid derived level graph. Equivalently, the first Betti number of each parity subgraph is non-increasing along any sequence of bridge switches.

Definition 4.6 (Intertwining tree)**.** A maximal planar graph G is an *intertwining tree* if its vertex set can be partitioned into two sets A and B such that both induced subgraphs $G[A]$ and $G[B]$ are trees.

Theorem 4.7. *A maximal planar graph G is an intertwining tree if and only if its dual G^* has a Hamiltonian cycle.*

Proof. ($\Rightarrow$) Let $V(G) = A \sqcup B$ with $G[A]$ and $G[B]$ trees. Every triangular face $\{x, y, z\}$ of G meets both A and B: if all three vertices were in A the triangle would be a cycle in the tree $G[A]$, and likewise for B. Draw a closed curve through the faces of G separating the A-vertices from the B-vertices within each face. Since every face is split, the curve visits every face exactly once and crosses an edge of G precisely when that edge joins A to B; it is therefore a Hamiltonian cycle of G^*.

(⇐) Let H be a Hamiltonian cycle of G^*. Drawn in the plane, H is a Jordan curve visiting every face of G once; let A and B be the vertices of G interior and exterior to H. The $2n-4$ edges of H cross exactly the edges of G between A and B, leaving $(3n-6)-(2n-4)=n-2$ edges inside $G[A]$ and $G[B]$ together. The edges inside A lie in the disk bounded by H and span A without enclosing a face (each face is cut by H), so $G[A]$ is a tree; likewise $G[B]$. □

Conjecture 4.8. Every maximal planar graph is a bridge-derived level graph of some Even Level Graph, an intertwining tree, or both.

Since a bridge-derived level graph is automatically a valid derived level graph, this is a stronger statement than the corresponding conjecture phrased with arbitrary E/O switches; it is also the form that the evidence below actually supports.

By Theorem 4.7, the intertwining-tree disjunct fails for G exactly when G^* is a counterexample to Tait's conjecture. The smallest such G^* have 38 vertices (Holton–McKay [1], exactly 6 graphs), so the smallest triangulations that are not intertwining trees occur at $n=21$ and there are exactly 6 of them. Below $n=21$ every maximal planar graph is an intertwining tree, which is why the disjunction holds trivially in that range.

Empirical status. For each isomorphism class of maximal planar graphs on n vertices, we ask whether (i) some isomorphic representative is a bridge-derived level graph of some Even Level Graph, and/or (ii) it is an intertwining tree. The conjecture holds for the class iff at least one of (i), (ii) holds. Below $n=21$ condition (ii) holds for *every* class, so the table mainly records how far the bridge-derived disjunct (i) reaches on its own. We classified bridge-derivability exhaustively for $n \le 9$, where every backward bridge-orbit can be enumerated in full.

n	# iso	bridge only	inter. only	both	missing	status
6	2	0	0	2	0	holds
7	5	0	1	4	0	holds
8	14	0	2	12	0	holds
9	50	0	14	36	0	holds

Here "bridge only" counts classes that are bridge-derived but not intertwining trees, "inter. only" the reverse, and "both" the intersection; "missing" counts classes that are neither (a counterexample). The "bridge only" column is 0 throughout this range precisely because every class is an intertwining tree for $n \le 20$; the "inter. only" counts $(1,2,14)$ are the classes that the bridge-derived disjunct alone does not yet reach, showing that bridge-derivability is strictly weaker than "intertwining tree" here and that the two disjuncts genuinely complement one another.

The boundary case $n=21$. The first triangulations that are *not* intertwining trees are the six duals of the Holton–McKay graphs, at $n=21$. For the disjunction to survive at $n=21$, each of these six must be a valid derived level graph. We find:

- All six duals are confirmed not intertwining trees (exhaustive check of all $2^{20}-1$ vertex bipartitions), consistent with Theorem 4.7.
- Two of the six are themselves Even Level Graphs (for a suitable source vertex), hence trivially valid derived level graphs. So the disjunction holds for them through the derived-level-graph disjunct – the first instances where that disjunct does work the intertwining-tree disjunct cannot.

- The remaining four are not Even Level Graphs for any source, and their full E/O-orbits ($\sim 10^8$ states per source labelling) are far too large to exhaust. Restricting to *bridge switches* (Definition 4.4) shrinks the relevant orbits by roughly two orders of magnitude and, crucially, keeps every reachable triangulation valid. A backward bridge-switch search over the valid parity partitions found an Even Level Graph witness for each of the four, so all four are *bridge-derived level graphs* (Definition 4.5) and hence valid derived level graphs. The witnessing orbits are small – between a few hundred and $\sim 1.7 \times 10^5$ states – even though other parity partitions of the same triangulations have orbits exceeding 10^6; finding one good partition suffices. Each witness is in fact only a *handful* of bridge switches from its dual: the explicit Even Level Graph, parity labelling, and bridge-switch sequence are recorded for all six – path lengths $3, 1, 2, 4$ for these four and 0 for the two that are Even Level Graphs outright – and each step has been verified to be a valid bridge switch.

Thus at $n = 21$ the disjunction is confirmed for all six critical iso classes: two are Even Level Graphs outright, and the other four are bridge-derived level graphs. The bridge-switch restriction is what made the search tractable – it both shrinks the orbit and guarantees validity, so any Even Level Graph located in a backward orbit is an immediate witness. Table 1 records the outcome for each dual.

dual	intertwining tree	Even Level Graph source	bridge switches to ELG
0	no	–	3
1	no	10	0
2	no	9	0
3	no	–	1
4	no	–	2
5	no	–	4

TABLE 1. The six Holton–McKay duals at $n = 21$, the first triangulations that are not intertwining trees. Each is a bridge-derived level graph: duals 1 and 2 are Even Level Graphs outright (zero switches), and the remaining four reach an Even Level Graph in 1–4 bridge switches. All witnesses are step-verified.

References

[1] D. A. Holton and B. D. McKay. *The smallest non-Hamiltonian 3-connected cubic planar graphs have 38 vertices.* Journal of Combinatorial Theory, Series B, 45(3):305–319, 1988.

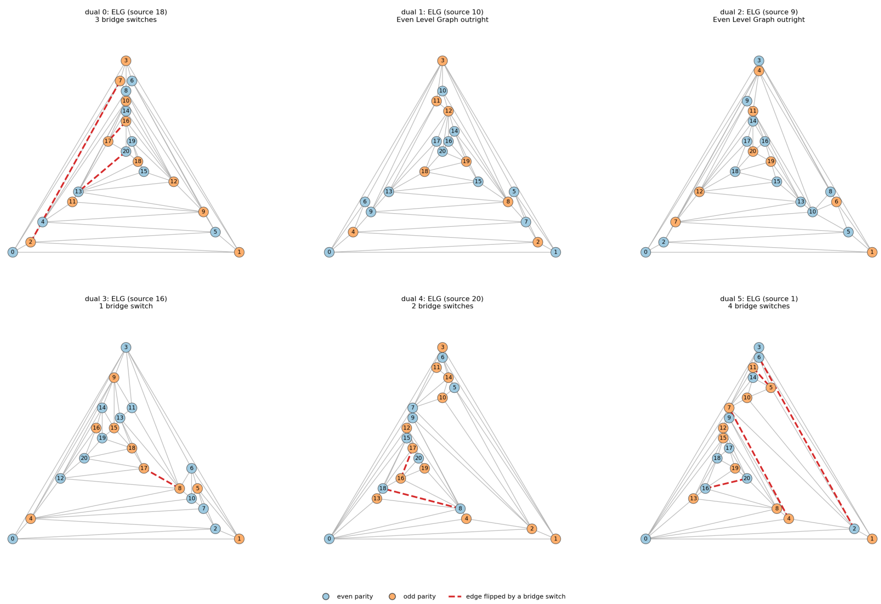

Figure 5. The witness Even Level Graph for each of the six Holton–McKay duals, drawn as a crossing-free planar graph and coloured by parity (blue even, orange odd, with respect to the fixed level-parity labelling). The dashed red edges are the same-parity edges that the bridge switches flip; flipping them yields the corresponding dual in Figure 6. Duals 1 and 2 are Even Level Graphs outright, so no edge is flipped.

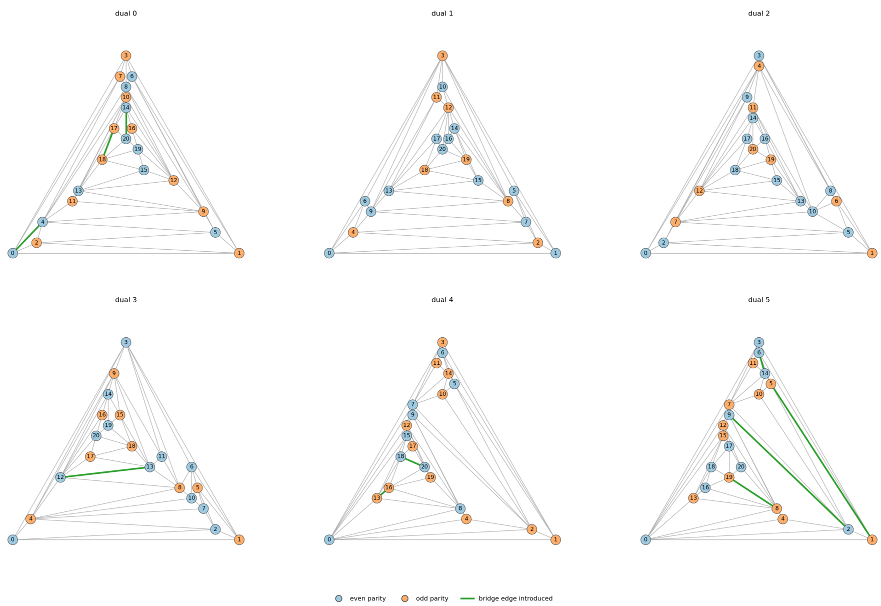

Figure 6. The six Holton–McKay duals, drawn as crossing-free planar graphs with the same parity colouring. The solid green edges are the bridge edges introduced by the switches from the Even Level Graphs of Figure 5. Each green edge is a bridge of its parity subgraph, so no new cycle – and in particular no odd cycle – is created; duals 1 and 2 coincide with their Even Level Graphs and have no added edge.