# EVEN LEVEL GRAPH GENERATORS

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Abstract.

## 1. Introduction

## 2. Definitions

Throughout, $G = (V, E)$ is a plane maximal planar graph (a triangulation) with a fixed planar embedding $\Pi_G$. We write $|V| = n$, so $|E| = 3n - 6$ and $G$ has $2n - 4$ triangular faces.

**Definition 2.1** (Level source)**.** A *level source* of $G$ is any vertex $v \in V$; we write $S = \{v\}$ for the level-0 source.

**Definition 2.2** (Levels)**.** Given a level source $S \subseteq V$, the *level* of $v \in V$ is $\ell_G(v) = \text{dist}_G(v, S)$, the graph distance from $v$ to the nearest source vertex.

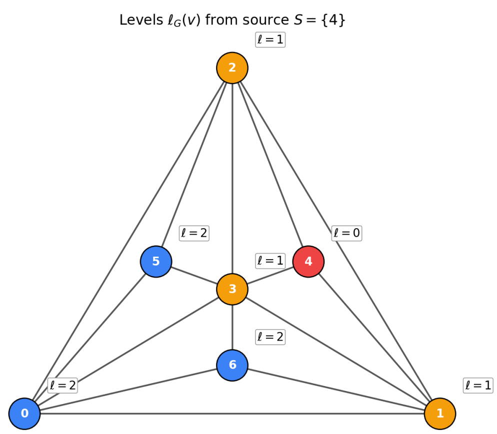


Figure 1. BFS levels from the degree-3 vertex source $S = \{4\}$. The source is level 0, its three neighbours are level 1, and the remaining vertices are level 2. Colour encodes the level.

**Definition 2.3** (Level cycle)**.** A *level cycle* of $G$ (with respect to a level source $S$) is a simple cycle in $G$ all of whose vertices have the same level.

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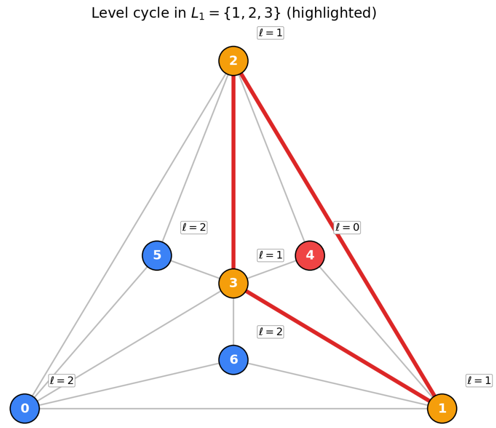


Figure 2. A level cycle in the triangulation of Figure 1. The triangle 1–2–3 is a simple cycle whose three vertices all lie at level 1, so it is a level cycle at level 1.

**Definition 2.4** (Edge switch)**.** Let $G$ be a triangulation with level source $S$, and let $e = uv$ be an edge of a level cycle of $G$. The *edge switch* at $e$ is the edge flip on $e$: writing $uvw$ and $uvx$ for the two triangular faces of $G$ containing $e$, the edge $uv$ is removed and the edge $wx$ is added. As with any edge flip, the result is a triangulation on the same vertex set provided $w$ and $x$ are non-adjacent in $G$.

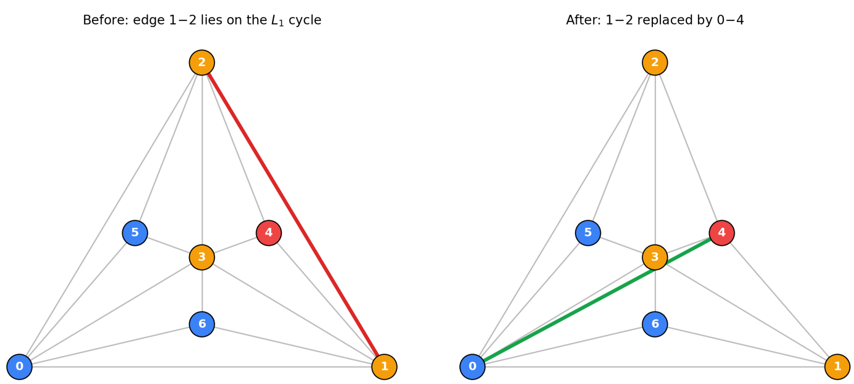


Figure 3. An edge switch on the level cycle of Figure 2. The chosen cycle edge $1 - 2$ is shared by the triangular faces $(0, 1, 2)$ and $(1, 2, 4)$; the switch deletes $1 - 2$ (red, left) and inserts $0 - 4$ (green, right). Vertex colours indicate the original levels in $G$.

**Definition 2.5** (Parity subgraph)**.** Let $G$ be a triangulation with level source $S$, and let $G'$ be a triangulation on the same vertex set as $G$. The *even parity subgraph* $E_{G,S}(G')$ is the subgraph of $G'$ induced by $\{v \in V : \ell_G(v) \equiv 0 \pmod 2\}$. The *odd parity subgraph* is defined analogously for odd $\ell_G$.

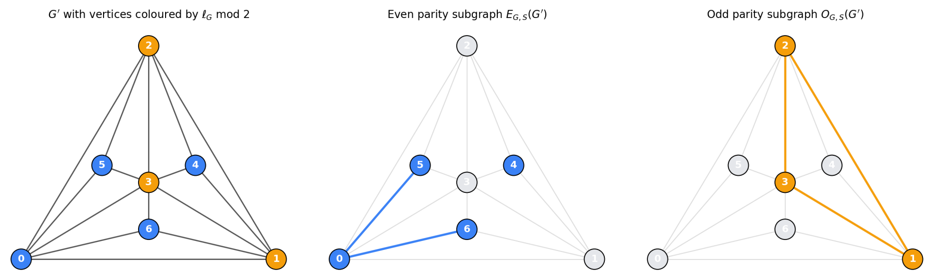


FIGURE 4. Parity subgraphs of $G' = T$ with respect to the level structure of Figure 1 (here we take $G = G' = T$). Left: $T$ with vertices coloured by $\ell_G$ mod 2 (blue = even, orange = odd). Middle: the even parity subgraph $E_{G,S}(G')$, induced on $\{0, 4, 5, 6\}$; only edges with both endpoints even appear. Right: the odd parity subgraph $O_{G,S}(G')$, induced on $\{1, 2, 3\}$; the highlighted triangle shows that $O_{G,S}(G')$ is not bipartite for this choice of $G'$.

## 3. Outerplanarity of level components

For each integer $k \geq 0$ and each $(G, S)$, write $L_k$ for the subgraph of $G$ induced by the level-$k$ vertices. A *level component* of $G$ (with respect to $S$) is a connected component of some $L_k$.

**Theorem 3.1.** *For every plane triangulation $G$ and every level source $S$ of $G$, every level component of $G$ is outerplanar.*

*Proof.* Since every subgraph of an outerplanar graph is outerplanar, it suffices to show that each level subgraph $L_k$ is outerplanar. For $k = 0$, $L_0 = S$ is a single vertex and is trivially outerplanar.

Fix $k \geq 1$ and let $D_k$ be the drawing of $L_k$ inherited from $\Pi_G$. Let $F^*$ be the face of $D_k$ containing the source. Suppose for contradiction that some $u \in L_k$ does not lie on $\partial F^*$, so $u$ lies on the boundary of some other face of $D_k$. Take any path $P$ in $G$ from $v_0 \in S$ to $u$. As a curve in $\Pi_G$, $P$ starts in $F^*$ and ends at a point off $\partial F^*$, so it must transition from $F^*$ to a different face of $D_k$; in a planar embedding this can happen only at a vertex of $D_k$, that is, at a level-$k$ vertex $w$ on $P$. Either $w \neq u$ (so $P$ has length $\geq \mathrm{dist}_G(S, w) + 1 \geq k + 1$), or $w = u$ (contradicting $u \notin \partial F^*$). Since every $S$-to-$u$ path has length $\geq k + 1$, $\mathrm{dist}_G(S, u) \geq k + 1$, contradicting $u \in L_k$. □

## 4. Even Level Graphs

**Definition 4.1** (Even Level Graph)**.** A plane triangulation $G$ with level source $S$ is an *Even Level Graph* if every level cycle of $G$ has even length.

**Theorem 4.2.** *Every Even Level Graph is 4-colorable.*

*Proof.* Since adjacent vertices in $G$ have levels differing by at most 1, any edge between two same-parity endpoints in fact connects two vertices at the same level. Hence

$$E_{G,S}(G) = \bigsqcup_{i \geq 0} L_{2i}, \qquad O_{G,S}(G) = \bigsqcup_{i \geq 0} L_{2i+1},$$

and each $L_k$ is bipartite because its cycles are level cycles of $G$, which have even length by hypothesis. Choose a 2-coloring of $E_{G,S}(G)$ in $\{\text{red}, \text{blue}\}$ and a 2-coloring of $O_{G,S}(G)$ in $\{\text{yellow}, \text{green}\}$. Same-parity edges of $G$ are properly colored by the respective bipartition; opposite-parity edges connect $\{\text{red}, \text{blue}\}$ to $\{\text{yellow}, \text{green}\}$. The combined assignment is a proper 4-coloring of $G$. □

**Definition 4.3** (Derived level graph)**.** Let $G$ be an Even Level Graph with level source $S$, and let $E$ and $O$ denote the edge sets of the even and odd parity subgraphs $E_{G,S}(G)$ and $O_{G,S}(G)$. A *derived level graph* of $G$ is a triangulation $G'$ on the same vertex set as $G$ obtained by a sequence of edge switches (Definition 2.4), each acting on an edge of $E$ or of $O$. We do not update $E$ or $O$ to reflect the level structure of intermediate triangulations: throughout the sequence, an edge is classified as belonging to $E$ (resp. $O$) if and only if both of its endpoints have even (resp. odd) level in $G$.

A derived level graph $G'$ is *valid* if both $E_{G,S}(G')$ and $O_{G,S}(G')$ contain only even cycles.

**Definition 4.4** (Bridge switch)**.** Let $G'$ be a triangulation reached from an Even Level Graph $G$, with parity classes inherited from $G$ as in Definition 4.3. An edge switch on an edge $e \in E \cup O$ of $G'$, replacing $uvw, uvx$ by the edge $wx$, is a *bridge switch* if either

- the new edge $wx$ is a cross-parity edge (one endpoint even, the other odd), so $wx$ enters neither parity subgraph; or
- $wx$ is a same-parity edge and is a *bridge* in the parity subgraph it joins – that is, $w$ and $x$ lie in different connected components of that parity subgraph, so adding $wx$ creates no new cycle.

**Definition 4.5** (Bridge-derived level graph)**.** A *bridge-derived level graph* of an Even Level Graph $G$ is a triangulation obtained from $G$ by a sequence of bridge switches (Definition 4.4).

Because a bridge switch never closes a cycle in a parity subgraph, it never introduces an odd cycle there. As an Even Level Graph has bipartite parity subgraphs (every level cycle is even), every bridge-derived level graph has bipartite parity subgraphs as well, and so is automatically a valid derived level graph. Equivalently, the first Betti number of each parity subgraph is non-increasing along any sequence of bridge switches.

**Definition 4.6** (Intertwining tree)**.** A maximal planar graph $G$ is an *intertwining tree* if its vertex set can be partitioned into two sets $A$ and $B$ such that both induced subgraphs $G[A]$ and $G[B]$ are trees.

**Theorem 4.7.** *A maximal planar graph $G$ is an intertwining tree if and only if its dual $G^*$ has a Hamiltonian cycle.*

*Proof.* (⇒) Let $V(G) = A \sqcup B$ with $G[A]$ and $G[B]$ trees. Every triangular face $\{x, y, z\}$ of $G$ meets both $A$ and $B$: if all three vertices were in $A$ the triangle would be a cycle in the tree $G[A]$, and likewise for $B$. Draw a closed curve through the faces of $G$ separating the $A$-vertices from the $B$-vertices within each face. Since every face is split, the curve visits every face exactly once and crosses an edge of $G$ precisely when that edge joins $A$ to $B$; it is therefore a Hamiltonian cycle of $G^*$.

($\Leftarrow$) Let $H$ be a Hamiltonian cycle of $G^*$. Drawn in the plane, $H$ is a Jordan curve visiting every face of $G$ once; let $A$ and $B$ be the vertices of $G$ interior and exterior to $H$. The $2n-4$ edges of $H$ cross exactly the edges of $G$ between $A$ and $B$, leaving $(3n-6)-(2n-4) = n-2$ edges inside $G[A]$ and $G[B]$ together. The edges inside $A$ lie in the disk bounded by $H$ and span $A$ without enclosing a face (each face is cut by $H$), so $G[A]$ is a tree; likewise $G[B]$. □

**Conjecture 4.8.** Every maximal planar graph is a bridge-derived level graph of some Even Level Graph, an intertwining tree, or both.

Since a bridge-derived level graph is automatically a valid derived level graph, this is a stronger statement than the corresponding conjecture phrased with arbitrary $E/O$ switches; it is also the form that the evidence below actually supports.

By Theorem 4.7, the intertwining-tree disjunct fails for $G$ exactly when $G^*$ is a counterexample to Tait's conjecture. The smallest such $G^*$ have 38 vertices (Holton–McKay [1], exactly 6 graphs), so the smallest triangulations that are not intertwining trees occur at $n = 21$ and there are exactly 6 of them. Below $n = 21$ every maximal planar graph is an intertwining tree, which is why the disjunction holds trivially in that range.

**The boundary case $n = 21$.** The first triangulations that are *not* intertwining trees are the six duals of the Holton–McKay graphs, at $n = 21$. For the disjunction to survive at $n = 21$, each of these six must be a valid derived level graph. We find:

- All six duals are confirmed not intertwining trees (exhaustive check of all $2^{20} - 1$ vertex bipartitions), consistent with Theorem 4.7.
- Two of the six are themselves Even Level Graphs (for a suitable source vertex), hence trivially valid derived level graphs. So the disjunction holds for them through the derived-level-graph disjunct – the first instances where that disjunct does work the intertwining-tree disjunct cannot.
- The remaining four are not Even Level Graphs for any source, and their full $E/O$-orbits ($\sim 10^8$ states per source labelling) are far too large to exhaust. Restricting to *bridge switches* (Definition 4.4) shrinks the relevant orbits by roughly two orders of magnitude and, crucially, keeps every reachable triangulation valid. A backward bridge-switch search over the valid parity partitions found an Even Level Graph witness for each of the four, so all four are *bridge-derived level graphs* (Definition 4.5) and hence valid derived level graphs. The witnessing orbits are small – between a few hundred and $\sim 1.7 \times 10^5$ states – even though other parity partitions of the same triangulations have orbits exceeding $10^6$; finding one good partition suffices. Each witness is in fact only a *handful* of bridge switches from its dual: the explicit Even Level Graph, parity labelling, and bridge-switch sequence are recorded for all six – path lengths $3, 1, 2, 4$ for these four and 0 for the two that are Even Level Graphs outright – and each step has been verified to be a valid bridge switch.

Thus at $n = 21$ the disjunction is confirmed for all six critical iso classes: two are Even Level Graphs outright, and the other four are bridge-derived level graphs. The bridge-switch restriction is what made the search tractable – it both shrinks the orbit and guarantees validity, so any Even Level Graph located in a backward orbit is an immediate witness. Table 1 records the outcome for each dual.

| dual | intertwining tree | Even Level Graph source | bridge switches to ELG |
|---|---|---|---|
| 0 | no | – | 3 |
| 1 | no | 10 | 0 |
| 2 | no | 9 | 0 |
| 3 | no | – | 1 |
| 4 | no | – | 2 |
| 5 | no | – | 4 |

TABLE 1. The six Holton–McKay duals at $n = 21$, the first triangulations that are not intertwining trees. Each is a bridge-derived level graph: duals 1 and 2 are Even Level Graphs outright (zero switches), and the remaining four reach an Even Level Graph in 1–4 bridge switches. All witnesses are step-verified.

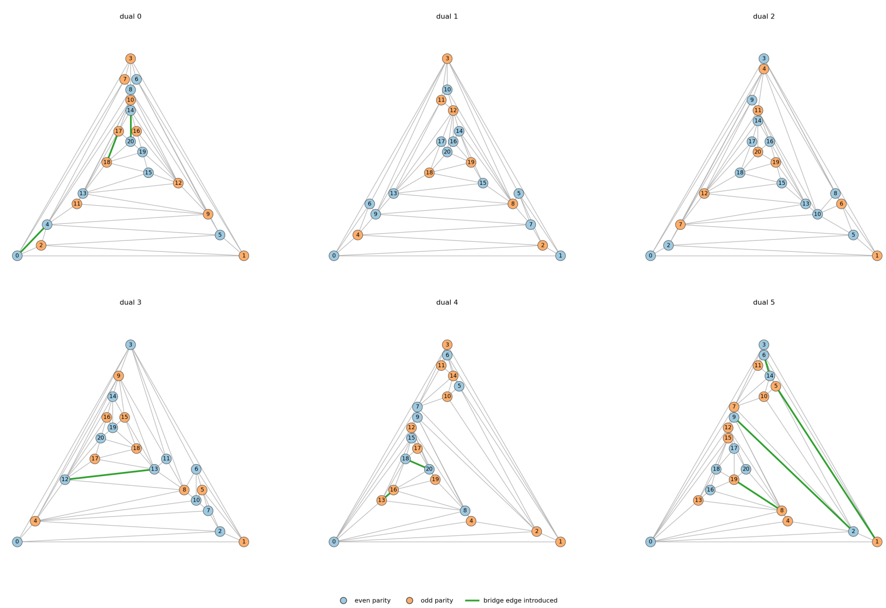


FIGURE 5. The six Holton–McKay duals, drawn as crossing-free planar graphs and coloured by parity (blue even, orange odd, with respect to the fixed level-parity labelling). The solid green edges are the bridge edges introduced by the bridge switches from each dual's witness Even Level Graph. Each green edge is a bridge of its parity subgraph, so no new cycle – and in particular no odd cycle – is created; duals 1 and 2 coincide with their Even Level Graphs and have no added edge.

## REFERENCES

[1] D. A. Holton and B. D. McKay. *The smallest non-Hamiltonian 3-connected cubic planar graphs have 38 vertices.* Journal of Combinatorial Theory, Series B, 45(3):305–319, 1988.