MEDIAL TIRE CUTS

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Abstract. Starting from the medial tire decomposition of a plane triangulation, we study the cuts that medial tires make in the full medial graph. We will show how to use medial tires to decompose the medial graph into a tree of three faces.

1. Introduction

This paper builds on the medial tire decomposition of [1]. For a plane triangulation G with fixed embedding we use freely the terminology and notation introduced there: the full medial graph $M(G)$, its decomposition into full medial tire graphs $\mathsf{M}(T)$ indexed by the treads T of the tire tree $\mathcal{T}(G,S)$ at a level source S, the annular medial cycle $A(T)$, and the boundary medial vertex sets.

We will show how to use medial tires to decompose the medial graph into a tree of three faces.

2. Cutting a full medial tire graph

We first describe a procedure that simultaneously *labels* and *cuts* a single full medial tire graph $\mathsf{M}(T)$ so that, after the cuts, the only faces are the outer face and 3-faces (triangles)—the teeth of [1]. The labelling assigns to each tooth an integer *walk depth*; the cuts break the cyclic adjacencies of the teeth so that what remains is a tree of 3-faces.

By a *cut* we mean the duplication of a single vertex of $\mathsf{M}(T)$: the vertex is split into two copies and the embedding is slit open along it (a planar unzip), separating the faces that meet only at that vertex. A cut therefore reduces the number of bounded faces that are not teeth.

Throughout we use the teeth, up and down teeth, apexes, bites, the annular medial cycle $A(T)$, and the auxiliary plane graph $B(T)$ of [1]. Each tooth is a 3-face of $\mathsf{M}(T)$, and the inner faces of $B(T)$ (the root face and the bite inner-gap faces) are the larger faces to be cut into teeth.

Definition 2.1 (Walk-depth labelling and cut)**.** Let $\mathsf{M}(T)$ be a full medial tire graph. Assign walk depths and cuts as follows.

1. Pick an arbitrary up tooth, the *entry tooth*. It has walk depth d.
2. Traverse all the teeth that bound the inner face incident to the entry tooth clockwise until we reach the entry tooth, incrementing the walk depth by 1 for each tooth traversed. (The *inner face incident to the entry tooth* is

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the inner face of $B(T)$ whose boundary contains the annular edge of $A(T)$ carrying the entry tooth.)

(3) When you reach the last tooth in the face, perform a *cut* by duplicating the annular vertex at which the traversal closes—the annular vertex of $A(T)$ shared by the last tooth and the entry tooth.

(4) Find the tooth t with the highest walk depth which is a member of a bite.

(5) If t is incident to a face F with unlabelled teeth, traverse the teeth in F starting from t in the direction of the tooth incident to t which is unlabelled, and increment the walk depth by 1 as you travel. (Here a tooth is *incident to t* when it shares an annular vertex of $A(T)$ with t.)

(6) Repeat steps (3)–(5) until all teeth have been labelled.

Remark 2.2 (Closing tooth of a descended face)*.* For the entry face the traversal of step (2) closes by returning to the entry tooth, so the cut of step (3) duplicates the annular vertex shared by the last tooth and the entry tooth. For a face F entered in step (5), the traversal instead closes upon reaching an already-labelled tooth: the other tooth of the bite through which F was entered. In both cases the cut of step (3) duplicates the annular vertex shared by the last newly labelled tooth and this *closing tooth*. Since both teeth of a bite are labelled while traversing its parent face, every descended face closes on such a tooth.

Example 2.3 (A worked walk-depth labelling and cut)**.** Figure 1 shows a full medial tire graph with annular cycle of length 8, generated by the full medial tire generator of [1]. Its eight teeth are: three up teeth on the annular edges $5, 6, 7$ in the root face; one bite pairing the annular edges 0 and 4; and three singleton down teeth on the annular edges $1, 2, 3$ lying in that bite's inner-gap face.

Take the up tooth on edge 5 as the entry tooth, with walk depth 0. Its inner face is the root face, bounded by the teeth on edges $5, 6, 7, 0, 4$ in clockwise order. Step (2) labels them

$$5 \mapsto 0, \quad 6 \mapsto 1, \quad 7 \mapsto 2, \quad 0 \mapsto 3, \quad 4 \mapsto 4,$$

and step (3) cuts by duplicating the annular vertex a_5 shared by the last tooth (edge 4) and the entry tooth (edge 5). The highest-depth bite tooth is now the one on edge 4 (walk depth 4); it is incident to the still-unlabelled inner-gap face of the bite $(0, 4)$. Entering that face from edge 4 toward its unlabelled neighbour, step (5) labels the three down teeth

$$3 \mapsto 5, \quad 2 \mapsto 6, \quad 1 \mapsto 7,$$

and closes on the already-labelled bite tooth of edge 0, so step (3) cuts by duplicating the annular vertex a_1 (Remark 2.2). All eight teeth are now labelled, and the two cuts leave only the outer face and the eight teeth as 3-faces. The labelling and cuts are produced by the script `experiments/medial_tire_cut_labelling.py`.

3. Chaining across the tire tree

Definition 2.1 labels and cuts a single full medial tire graph. We extend it to the whole medial graph $M(G)$ through the medial tire decomposition of [1]: the tire tree decomposes $M(G)$ into full medial tire graphs $\mathsf{M}(T)$, one per tread T, glued along their boundary medial vertices. A parent tread's inner level cycle is a child tread's outer level cycle, and the boundary medial vertices on that shared cycle belong to both treads.

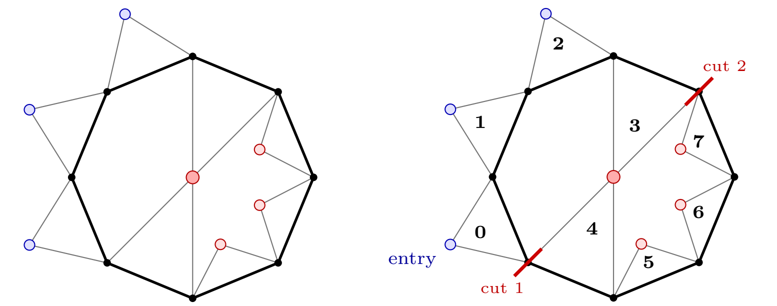

FIGURE 1. A full medial tire graph (left) and its walk-depth labelling and cut (right), from Example 2.3. Black vertices are the annular medial vertices of the cycle $A(T)$; blue vertices are up-tooth apexes, red vertices are down-tooth apexes, and the larger red vertex is the shared apex of the bite on annular edges 0 and 4. On the right, each tooth carries its walk depth, and the two red slits mark the cuts: *cut 1* duplicates a_5 as the root-face traversal closes, and *cut 2* duplicates a_1 as the bite's inner-gap face closes. After the cuts the only bounded faces are the eight teeth.

The key incidence is this. A *boundary* (singleton) down tooth of a parent tread and the up tooth of the child tread glued to it across the shared level cycle have the *same apex*: both apexes are the same medial vertex of $M(G)$, namely the medial vertex of an edge with both endpoints on the shared level cycle. We use this to carry the walk depth from a parent into its children.

We label tread by tread, outward from the root:

- a tread with no parent in the decomposition—in particular the innermost recognised tread—is treated as a *root* and entered at an arbitrary up tooth with walk depth 0;
- a child tread is entered at the up tooth whose apex is the parent's boundary down tooth of lowest walk depth; that entry up tooth's walk depth is one more than that down tooth's, and the walk then increments locally within the child as in Definition 2.1.

Remark 3.1 (Candidate down teeth for chaining). The down teeth eligible to fix a child's entry are exactly the *boundary* (singleton) down teeth of the parent: those lying in a single tread face, whose apex is the shared boundary medial vertex glued to a child up tooth. A bite's two down teeth are *not* eligible. By the definition of a bite in [1] its annular edge borders two tread faces, so a bite tooth is interior to the parent tread and its apex is a boundary medial vertex of no child. Hence "the down tooth of lowest walk depth" is read among the boundary down teeth only; a bite of even lower walk depth is skipped.

Applying every tread's cuts to $M(G)$ assembles the per-tread labellings and cuts into a single cut graph of $M(G)$ together with a global walk-depth label map. This pipeline—random maximal planar graph, medial graph, tire decomposition at a vertex level source, and chained walk-depth labelling and cut—is carried out by the experiment script `experiments/run_medial_tire_cut_experiment.py`.

Example 3.2 (A medial tire cut from a random graph)**.** Run on a random maximal planar graph on 20 vertices (seed 72, level source vertex 9), the experiment yields a single recognised tread T_2, drawn in Figure 2 with the walk-depth labelling and cut emitted by the graphics companion `experiments/draw_medial_tire_cut.py`. Its annular cycle has length 8, with up teeth on annular edges $0, 3, 4$, singleton down teeth on $1, 6, 7$, and a bite on the non-incident annular edges 2 and 5 (the central shared apex). Entering at the up tooth on edge 0 with walk depth 0, the root face is labelled in order $(0, 1, 2$ then $3, 4, 5)$ and *cut 1* duplicates a_0 as it closes; the walk then descends through the bite into its inner-gap face, labelling the two teeth there $(6, 7)$, and *cut 2* duplicates a_3 as that face closes. The two cuts leave only the outer face and the eight teeth as 3-faces.

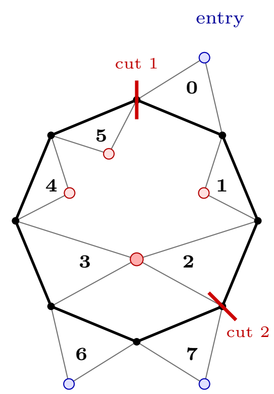

Figure 2. The recognised tread T_2 of the medial tire decomposition of a random maximal planar graph on 20 vertices (Example 3.2), with its walk-depth labelling and cut. Black vertices are the annular medial vertices of $A(T)$; blue vertices are up-tooth apexes and red vertices down-tooth apexes, the larger red vertex being the shared apex of the bite on annular edges 2 and 5. Each tooth carries its walk depth; the red slits are the two cuts.

References

[1] E. Bauerfeld, *Medial Tire Decompositions of Plane Triangulations*, manuscript (math-research repository), 2026.