EVEN LEVEL GRAPH GENERATORS

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Abstract.

1. Introduction

2. Definitions

Throughout, $G = (V, E)$ is a plane maximal planar graph (a triangulation) with a fixed planar embedding Π_G. We write $|V| = n$, so $|E| = 3n - 6$ and G has $2n - 4$ triangular faces.

Definition 2.1 (Level source)**.** A *level source* of G is any vertex $v \in V$; we write $S = \{v\}$ for the level-0 source.

Definition 2.2 (Levels)**.** Given a level source $S \subseteq V$, the *level* of $v \in V$ is $\ell_G(v) = \mathrm{dist}_G(v, S)$, the graph distance from v to the nearest source vertex.

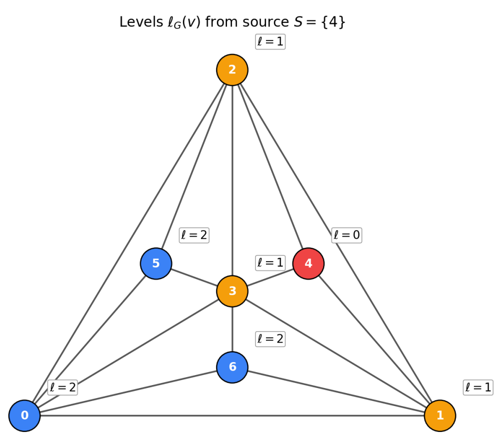

Figure 1. BFS levels from the degree-3 vertex source $S = \{4\}$. The source is level 0, its three neighbours are level 1, and the remaining vertices are level 2. Colour encodes the level.

Definition 2.3 (Level cycle)**.** A *level cycle* of G (with respect to a level source S) is a simple cycle in G all of whose vertices have the same level.

2010 *Mathematics Subject Classification.* Primary .

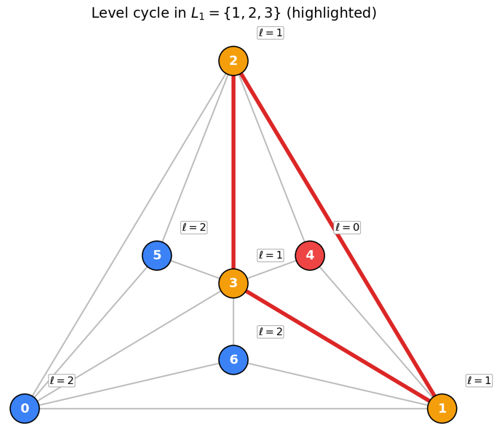

Figure 2. A level cycle in the triangulation of Figure 1. The triangle 1–2–3 is a simple cycle whose three vertices all lie at level 1, so it is a level cycle at level 1.

Definition 2.4 (Edge switch)**.** Let G be a triangulation with level source S, and let $e = uv$ be an edge of a level cycle of G. The *edge switch* at e is the edge flip on e: writing uvw and uvx for the two triangular faces of G containing e, the edge uv is removed and the edge wx is added. As with any edge flip, the result is a triangulation on the same vertex set provided w and x are non-adjacent in G.

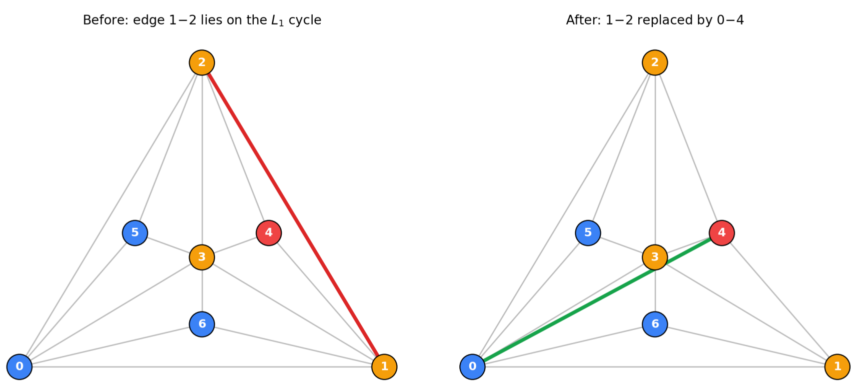

Figure 3. An edge switch on the level cycle of Figure 2. The chosen cycle edge $1-2$ is shared by the triangular faces $(0, 1, 2)$ and $(1, 2, 4)$; the switch deletes $1-2$ (red, left) and inserts $0-4$ (green, right). Vertex colours indicate the original levels in G.

Definition 2.5 (Parity subgraph)**.** Let G be a triangulation with level source S, and let G' be a triangulation on the same vertex set as G. The *even parity subgraph* $E_{G,S}(G')$ is the subgraph of G' induced by $\{v \in V : \ell_G(v) \equiv 0 \pmod 2\}$. The *odd parity subgraph* is defined analogously for odd ℓ_G.

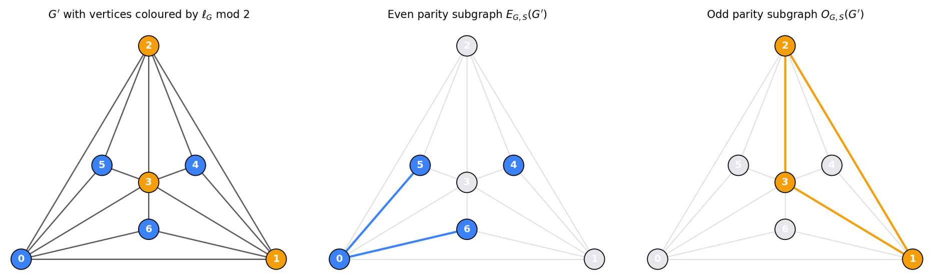

FIGURE 4. Parity subgraphs of $G' = T$ with respect to the level structure of Figure 1 (here we take $G = G' = T$). Left: T with vertices coloured by ℓ_G mod 2 (blue = even, orange = odd). Middle: the even parity subgraph $E_{G,S}(G')$, induced on $\{0, 4, 5, 6\}$; only edges with both endpoints even appear. Right: the odd parity subgraph $O_{G,S}(G')$, induced on $\{1, 2, 3\}$; the highlighted triangle shows that $O_{G,S}(G')$ is not bipartite for this choice of G'.

3. Outerplanarity of level components

For each integer $k \geq 0$ and each (G, S), write L_k for the subgraph of G induced by the level-k vertices. A *level component* of G (with respect to S) is a connected component of some L_k.

Theorem 3.1. *For every plane triangulation G and every level source S of G, every level component of G is outerplanar.*

Proof. Since every subgraph of an outerplanar graph is outerplanar, it suffices to show that each level subgraph L_k is outerplanar. For $k = 0$, $L_0 = S$ is a single vertex and is trivially outerplanar.

Fix $k \geq 1$ and let D_k be the drawing of L_k inherited from Π_G. Let F^* be the face of D_k containing the source. Suppose for contradiction that some $u \in L_k$ does not lie on ∂F^*, so u lies on the boundary of some other face of D_k. Take any path P in G from $v_0 \in S$ to u. As a curve in Π_G, P starts in F^* and ends at a point off ∂F^*, so it must transition from F^* to a different face of D_k; in a planar embedding this can happen only at a vertex of D_k, that is, at a level-k vertex w on P. Either $w \neq u$ (so P has length $\geq \mathrm{dist}_G(S, w) + 1 \geq k + 1$), or $w = u$ (contradicting $u \notin \partial F^*$). Since every S-to-u path has length $\geq k + 1$, $\mathrm{dist}_G(S, u) \geq k + 1$, contradicting $u \in L_k$. □

4. Even Level Graphs

Definition 4.1 (Even Level Graph)**.** A plane triangulation G with level source S is an *Even Level Graph* if every level cycle of G has even length.

Theorem 4.2. *Every Even Level Graph is* 4*-colorable.*

Proof. Since adjacent vertices in G have levels differing by at most 1, any edge between two same-parity endpoints in fact connects two vertices at the same level. Hence

$$E_{G,S}(G) = \bigsqcup_{i \geq 0} L_{2i}, \qquad O_{G,S}(G) = \bigsqcup_{i \geq 0} L_{2i+1},$$

and each L_k is bipartite because its cycles are level cycles of G, which have even length by hypothesis. Choose a 2-coloring of $E_{G,S}(G)$ in $\{\text{red}, \text{blue}\}$ and a 2-coloring of $O_{G,S}(G)$ in $\{\text{yellow}, \text{green}\}$. Same-parity edges of G are properly colored by the respective bipartition; opposite-parity edges connect $\{\text{red}, \text{blue}\}$ to $\{\text{yellow}, \text{green}\}$. The combined assignment is a proper 4-coloring of G. □

Definition 4.3 (Derived level graph)**.** Let G be an Even Level Graph with level source S, and let E and O denote the edge sets of the even and odd parity subgraphs $E_{G,S}(G)$ and $O_{G,S}(G)$. A *derived level graph* of G is a triangulation G' on the same vertex set as G obtained by a sequence of edge switches (Definition 2.4), each acting on an edge of E or of O. We do not update E or O to reflect the level structure of intermediate triangulations: throughout the sequence, an edge is classified as belonging to E (resp. O) if and only if both of its endpoints have even (resp. odd) level in G.

A derived level graph G' is *valid* if both $E_{G,S}(G')$ and $O_{G,S}(G')$ contain only even cycles.

Definition 4.4 (Intertwining tree)**.** A maximal planar graph G is an *intertwining tree* if its vertex set can be partitioned into two sets A and B such that both induced subgraphs $G[A]$ and $G[B]$ are trees.

Theorem 4.5. *A maximal planar graph G is an intertwining tree if and only if its dual G^* has a Hamiltonian cycle.*

Proof. ($\Rightarrow$) Let $V(G) = A \sqcup B$ with $G[A]$ and $G[B]$ trees. Every triangular face $\{x, y, z\}$ of G meets both A and B: if all three vertices were in A the triangle would be a cycle in the tree $G[A]$, and likewise for B. Draw a closed curve through the faces of G separating the A-vertices from the B-vertices within each face. Since every face is split, the curve visits every face exactly once and crosses an edge of G precisely when that edge joins A to B; it is therefore a Hamiltonian cycle of G^*.

($\Leftarrow$) Let H be a Hamiltonian cycle of G^*. Drawn in the plane, H is a Jordan curve visiting every face of G once; let A and B be the vertices of G interior and exterior to H. The $2n-4$ edges of H cross exactly the edges of G between A and B, leaving $(3n-6)-(2n-4) = n-2$ edges inside $G[A]$ and $G[B]$ together. The edges inside A lie in the disk bounded by H and span A without enclosing a face (each face is cut by H), so $G[A]$ is a tree; likewise $G[B]$. □

Conjecture 4.6. Every maximal planar graph is a valid derived level graph of some Even Level Graph, an intertwining tree, or both.

By Theorem 4.5, the intertwining-tree disjunct fails for G exactly when G^* is a counterexample to Tait's conjecture. The smallest such G^* have 38 vertices (Holton–McKay [1], exactly 6 graphs), so the smallest triangulations that are not intertwining trees occur at $n = 21$ and there are exactly 6 of them. Below $n = 21$ every maximal planar graph is an intertwining tree, which is why the disjunction holds trivially in that range.

Empirical status. For each isomorphism class of maximal planar graphs on n vertices, we ask whether (i) some isomorphic representative is reachable from some Even Level Graph via E/O-edge switches ("derived"), and/or (ii) it is an intertwining tree. The conjecture holds for the class iff at least one of (i), (ii) holds.

n	# iso	derived only	inter. only	both	missing	status
6	2	0	0	2	0	holds
7	5	0	0	5	0	holds
8	14	0	0	14	0	holds
9	50	0	1	49	0	holds
10	233	0	0	233	0	holds
11	1249	0	0	1249	0	holds
12	7595	0	1	7594	0	holds

The boundary case $n = 21$. The first triangulations that are *not* intertwining trees are the six duals of the Holton–McKay graphs, at $n = 21$. For the disjunction to survive at $n = 21$, each of these six must be a valid derived level graph. We find:

- All six duals are confirmed not intertwining trees (exhaustive check of all $2^{20} - 1$ vertex bipartitions), consistent with Theorem 4.5.
- Two of the six are themselves Even Level Graphs (for a suitable source vertex), hence trivially valid derived level graphs. So the disjunction holds for them through the derived-level-graph disjunct – the first instances where that disjunct does work the intertwining-tree disjunct cannot.
- The remaining four are not Even Level Graphs for any source. A bounded backward E/O-orbit search (tens of thousands of states, a handful of source labellings) found no Even Level Graph in their orbits, but this is far too shallow relative to the orbit size at $n = 21$ to be conclusive; their status as derived level graphs is open.

Thus at $n = 21$ the disjunction is confirmed for two of the six critical iso classes and undetermined for the other four. Settling those four – equivalently, deciding E/O-orbit reachability from an Even Level Graph – is the first genuinely open instance of the conjecture, and calls for either a better reachability algorithm or a structural invariant of E/O-orbits.

References

[1] D. A. Holton and B. D. McKay. *The smallest non-Hamiltonian 3-connected cubic planar graphs have 38 vertices.* Journal of Combinatorial Theory, Series B, 45(3):305–319, 1988.