PLANE DEPTH

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Abstract. Given a plane embedding of a graph with outer cycle C, the *plane depth* of a vertex is its graph distance to C. We develop this depth function into a layered combinatorial structure on plane triangulations: the subgraph induced by each depth level is outerplanar (recovering Baker's notion of a k-outerplanar graph); each triangular face is classified by its depth multiset as *up*, *down*, or *neutral*; and the *deep embedding* of a maximal planar graph, obtained by inserting a vertex into every neutral face (including the outer face), has every face either up or down. Pairing adjacent triangles across their unique level edge yields a *quadrilateral decomposition* of the spherical deep embedding into three combinatorial types: shallow diamonds, deep diamonds, and S quads.

This paper isolates the foundational depth-and-decomposition material that supports several downstream applications — including the quadrilateral sequencing of [2] and the nested-tire colouring framework of [3].

1. Definitions

Definition 1.1. Let G be a graph with a plane embedding, and let C be the outer cycle of that embedding. The *plane depth* of a vertex $v \in V(G)$ relative to the embedding and C is

$$\mathrm{depth}(v) = \min_{u \in V(C)} d(v, u),$$

where $d(v, u)$ denotes the graph distance between v and u in G.

Definition 1.2. An edge $\{u, v\} \in E(G)$ is a *level edge* if $\mathrm{depth}(u) = \mathrm{depth}(v)$.

Definition 1.3. A triangle $\{u, v, w\}$ in G is an *up triangle* if the multiset of depths of its vertices is $\{d, d + 1, d + 1\}$ for some $d \geq 0$, a *down triangle* if the multiset of depths is $\{d, d, d + 1\}$ for some $d \geq 0$, and a *neutral triangle* if the multiset of depths is $\{d, d, d\}$ for some $d \geq 0$.

Remark 1.4. We now relate our terminology to existing terminology, namely k-outerplanar graphs [1]. The following definition and lemma show that the subgraph induced by any single depth level relative to any source set on the outer face is outerplanar, i.e. 1-outerplanar in the sense of Baker.

Definition 1.5. A plane graph is *outerplanar* if every vertex lies on the outer face. More generally, a plane graph is *k-outerplanar* for $k \geq 1$ if removing all vertices on the outer face yields a $(k - 1)$-outerplanar graph, where every graph on the empty vertex set is 0-outerplanar.

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2. Outerplanarity of depth levels

Lemma 2.1. *Let G be a planar graph with a plane embedding Π, and let $S \subseteq V(G)$ be a nonempty set of vertices, every one of which lies on the boundary of the outer face of Π. For each $d \geq 0$, the subgraph of G induced by*

$$V_d^S := \{v \in V(G) : \mathrm{dist}_G(v, S) = d\}$$

is outerplanar.

The special case $S = V(C)$, where C is the outer cycle, recovers $V_d^S = V_d$ (depth-d vertices as in Definition 1.1) and is the form most often used in applications.

Proof. Let $H = G[V_d^S]$ with the plane embedding inherited from Π. It suffices to show that every vertex of H lies on the outer face of H.

For $d = 0$, $V_0^S = S$, and by hypothesis every vertex of S lies on the boundary of the outer face of Π. Removing the vertices and edges of $G \setminus H$ from the embedding only enlarges or merges face regions, so the outer face of Π is contained in the outer face of H, and every vertex of S remains on the outer face of H.

For $d \geq 1$, let U be the open subset of the plane obtained by removing all vertices and edges of H. We show every $v \in V_d^S$ lies on the boundary of the component U_{out} of U containing the outer face of Π.

Since every vertex in $V_{<d}^S := \bigcup_{e<d} V_e^S$ has a shortest path to S passing entirely through $V_{<d}^S$, the subgraph $G[V_{<d}^S]$ is connected and contains S. Its vertices and edges lie in U (none belong to H), and S borders the outer face of Π, so $G[V_{<d}^S]$ and the outer face of Π are connected within U, hence both lie in U_{out}.

Now let $v \in V_d^S$. Since $d \geq 1$, there exists $u \in V_{d-1}^S$ adjacent to v in G. The edge $\{v, u\}$ is not in H, so it lies in U. Since $u \in V_{d-1}^S \subseteq U_{\mathrm{out}}$ and $\{v, u\}$ is a connected subset of U containing u, the entire edge lies in U_{out}. The vertex v is an endpoint of this edge but is not in U, so v lies on the boundary of U_{out}, i.e. on the outer face of H. $\square$

3. Deep embedding

Definition 3.1. Let G be a maximal planar graph with a plane embedding and outer cycle C. The *deep embedding* of G is the graph G' obtained from G by the following operation: for every neutral triangular face $\{u, v, w\}$ of G — *including the outer face*, whose vertices are the three vertices of C — add a new vertex x placed in that face and adjacent to each of u, v, and w. The vertex added inside the outer face is denoted x^* and called the *outer-cap vertex*; the three triangular faces it induces with the edges of C are the *outer-cap faces.* We henceforth view G' as embedded on the sphere S^2, with no distinguished outer face.

Lemma 3.2. *Let G' be the deep embedding of a maximal planar graph G. Every face of G' is either an up triangle or a down triangle.*

Proof. We first establish that for any edge $\{p, q\}$ in G, the depths of p and q differ by at most 1. Suppose for contradiction that $\mathrm{depth}(p) = d$ and $\mathrm{depth}(q) = d + n$ for some $n \geq 2$. Since $\mathrm{depth}(p) = d$, there exists a path of length d from p to some vertex of C. Prepending the edge $\{q, p\}$ gives a path of length $d + 1$ from q to C, so $\mathrm{depth}(q) \leq d+1 < d+n$, a contradiction. The case $\mathrm{depth}(q) = d - n$ is handled identically: there exists a path of length $d - n$ from q to some vertex of C, and prepending the edge $\{p, q\}$ gives a path of length $d - n + 1 \leq d - 1 < d$ from p to C, contradicting $\mathrm{depth}(p) = d$.

Since G is a triangulation, every interior face of G is a triangle $\{u, v, w\}$ with all three pairs adjacent. By the above, each pair of vertices in a triangle differs in depth by at most 1, so no triangle can contain vertices of depths d and $d + 2$ simultaneously. The possible depth patterns for a triangle in G are therefore exactly a neutral triangle, a down triangle, or an up triangle.

We now consider each case under the deep embedding.

Case 1: up triangle or down triangle. These triangles are not modified by the deep embedding, so they remain as faces of G', satisfying the lemma.

Case 2: neutral triangle. The deep embedding inserts a new vertex x adjacent to u, v, and w, replacing the face $\{u, v, w\}$ with three new faces $\{u, v, x\}$, $\{v, w, x\}$, and $\{u, w, x\}$. It remains to determine the depth of x in G'. Since x is adjacent only to u, v, and w, every path in G' from x to C must pass through one of them, so x has strictly greater depth than u, v, and w. Each of the three new faces is thus a down triangle, satisfying the lemma. The same argument applies to the outer face: the outer-cap vertex x^* is adjacent to all three vertices of C (which lie at depth 0), so $\text{depth}(x^*) = 1$, and each of the three outer-cap faces is a down triangle.

Since every face of G' falls into one of these cases, the result follows. □

4. Quadrilateral decomposition

Lemma 4.1. *Every interior face of G' has exactly one level edge.*

Proof. By Lemma 3.2, each interior face is an up triangle (depths $\{d, d+1, d+1\}$) or a down triangle (depths $\{d, d, d + 1\}$). In both cases, exactly one of the three vertex pairs has equal depth. □

Lemma 4.2. *Let $e = \{p, q\}$ be any level edge of G'. Then e is the unique level edge of both faces incident to it.*

Proof. On the sphere, both faces T, T' incident to e are triangles. Since p and q have equal depth, e is a level edge of T and of T', and by Lemma 4.1 each has e as its unique level edge. □

Definition 4.3. The *quadrilateral decomposition* of G' pairs each face of G' with the face on the other side of its (unique) level edge. Each pair, together with the four non-level edges of the two triangles, bounds a *quadrilateral* of the decomposition.

Remark 4.4. Because G' is taken on the sphere, every edge lies between two triangular faces, so the pairing above applies uniformly. In particular, each edge of C is a level edge shared between one interior boundary down triangle (depths $\{0, 0, 1\}$, with the depth-1 vertex inside C) and one outer-cap down triangle (depths $\{0, 0, 1\}$, with apex x^*). The three resulting quadrilaterals, one per edge of C, are the *boundary deep diamonds*; they are the outermost quadrilaterals of the decomposition.

Definition 4.5. Each quadrilateral is one of three types, classified by the depths of its two non-level vertices relative to the depth d of the shared level edge:

- a *shallow diamond*, formed by two up triangles, with vertex depths $(d - 1, d, d - 1, d)$ around the boundary;
- a *deep diamond*, formed by two down triangles, with vertex depths $(d + 1, d, d + 1, d)$ around the boundary;
- an *S quad*, formed by one up and one down triangle, with vertex depths $(d - 1, d, d + 1, d)$ around the boundary.

References

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