MEDIAL TIRE DECOMPOSITIONS OF PLANE TRIANGULATIONS

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Abstract. We use the nested tire decomposition of a plane triangulation to induce a decomposition of its full medial graph into medial tire subgraphs. For a plane triangulation G, the medial graph $M(G)$ is naturally isomorphic to the medial graph of the planar dual G^*, and proper 3-vertex-colourings of $M(G)$ are equivalent to proper 3-edge-colourings of the cubic dual. Thus Tait's reformulation of the Four Colour Theorem may be studied through proper vertex 3-colourings of medial subgraphs. We define medial tire pieces, their boundary-state restriction relations, and a chain-pigeonhole conjecture for compatible medial boundary states across the tire tree.

1. Introduction

A classical theorem of Tait recasts the Four Colour Theorem in dual, edge-colouring terms: a plane triangulation G is properly 4-vertex-colourable if and only if its dual cubic graph G^* is properly 3-edge-colourable. The present paper records a medial version of this viewpoint. The vertices of the medial graph $M(G)$ correspond to edges of G, and adjacency in $M(G)$ records consecutiveness of edges around vertices and faces of G. Since planar duality interchanges vertices and faces while preserving the edge set, $M(G)$ is naturally isomorphic to $M(G^*)$.

Consequently a proper vertex 3-colouring of $M(G)$ is the same object as a proper edge 3-colouring of G^*. This suggests another route toward the Four Colour Theorem: rather than colouring the dual cubic graph directly, decompose the full medial graph into local annular pieces and try to prove that their proper vertex 3-colouring boundary restrictions always compose.

The structural input is the nested tire decomposition of [1]. A level source in a plane triangulation determines a rooted tree of tire treads. Each tread is an annular triangulated region with an outer boundary, an inner outerplanar graph, and annular triangular faces. We show that this decomposition induces a decomposition of $M(G)$ into medial tire subgraphs. The boundary data of a medial tire are proper 3-colourings of the medial vertices corresponding to boundary edges in the associated dual tire graph.

2. Background

Throughout, G is a simple plane maximal planar graph with fixed embedding, and G^* denotes its full planar dual. We use the level source, dual depth, tire graph, tire tread, and tire-tree terminology of [1]. In particular, a level source

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S determines a rooted tire tree $\mathcal{T}(G,S)$ whose vertices are tire treads and whose parent-child relation records nested containment across level-cycle interfaces.

Definition 2.1 (Medial graph)**.** Let H be a plane graph. The *medial graph* $M(H)$ has one vertex m_e for each edge $e \in E(H)$. Two medial vertices m_e, m_f are adjacent whenever e and f are consecutive in the cyclic order of edges around a vertex of H or around a face of H. The embedding is the standard one obtained by placing m_e at the midpoint of e and drawing medial edges through the vertex- and face-corners of H.

Remark 2.2. If H has bridges or vertices of degree 1, the usual medial construction may create parallel edges or loops depending on the chosen convention. In this paper the main application is to plane triangulations and their cubic planar duals, where the medial graph is a loopless 4-regular plane graph.

Proposition 2.3 (Medial dual invariance)**.** *Let H be a connected plane graph and let H^* be its planar dual. Then there is a natural plane-graph isomorphism*

$$M(H) \cong M(H^*).$$

Proof. Each edge $e \in E(H)$ corresponds to a unique dual edge $e^* \in E(H^*)$, giving a bijection $m_e \mapsto m_{e^*}$ between the vertices of $M(H)$ and $M(H^*)$. In $M(H)$ two vertices m_e, m_f are adjacent exactly when e and f are consecutive around either a vertex or a face of H. Under duality, vertices and faces are interchanged, and the cyclic order of the corresponding dual edges around the dual face or dual vertex is the same up to reversal. Thus the same pairs are medial-adjacent in $M(H^*)$, and the midpoint construction identifies the two embedded medial graphs. □

Corollary 2.4 (Tait colourings as medial vertex colourings)**.** *Let G be a simple plane triangulation. Proper vertex 3-colourings of $M(G)$ are in natural bijection with proper 3-edge-colourings of the cubic planar dual G^*.*

Proof. By Proposition 2.3, $M(G) \cong M(G^*)$. Vertices of $M(G^*)$ correspond to edges of G^*, and two such vertices are adjacent exactly when the corresponding dual edges are incident and consecutive around a vertex or face of G^*. Since G^* is cubic, proper vertex 3-colouring of $M(G^*)$ is therefore equivalent to assigning three colours to the edges of G^* so that the three edges incident to each dual vertex receive pairwise distinct colours. □

3. Medial tire pieces

Definition 3.1 (Full medial tire graph)**.** Let T be a tire tread in the tire tree $\mathcal{T}(G,S)$ supplied by [1]. The *full medial tire graph* of T, denoted $\mathsf{M}(T)$, is the subgraph of $M(G)$ induced by the medial vertices m_e with e an edge of G incident to at least one triangular face in the tread T. The medial vertices corresponding to annular edges of T are called *annular medial vertices.*

Remark 3.2. In the ambient-triangulation setting, the full medial tire graph $\mathsf{M}(T)$ coincides with the omitted-edge medial tire graph studied in [1]. Indeed, the medial edges of $\mathsf{M}(T)$ are contributed by corners of annular triangular tread faces. Such a face contains at most one outer-boundary edge and at most one inner-boundary edge, so it does not contribute a medial edge between two outer-boundary edges or between two inner-boundary edges. Similarly, chords of the inner outerplanar graph lie outside the annular tread and are not incident to annular tread faces.

Thus the deletion rule used for the earlier reduced medial tire graph removes no edges from the ambient object $\mathsf{M}(T)$.

The distinction only appears in the standalone drawing convention where the outer and inner boundary walks are added as artificial faces before forming a medial graph. Those artificial faces create same-boundary medial edges, and the reduced construction deletes them.

Theorem 3.3 (Annular medial colour bound)**.** *Let $T = (B_{\text{out}}, O, E_{\text{ann}})$ be a tire tread with non-degenerate boundaries and simple inner boundary B_{in}. Let $A(T)$ be the subgraph of $\mathsf{M}(T)$ induced by the annular medial vertices. For a graph H, write $\mathrm{Col}_3(H)$ for the set of proper 3-vertex-colourings of H. Then $A(T)$ is a cycle—the* annular cycle *of T—and*

$$|\,\mathrm{Col}_3(\mathsf{M}(T))| \;\le\; |\,\mathrm{Col}_3(A(T))|.$$

Proof. Since the tread is a triangulated annulus with no vertices in its interior, each annular face has exactly one boundary edge, lying either on B_{out} or on B_{in}, and exactly two annular edges. As the annular faces are traversed cyclically around the tread, consecutive faces share one annular edge. Equivalently, the annular edges occur in a cyclic order in which each annular face contains two consecutive annular edges. Hence the subgraph of $\mathsf{M}(T)$ induced by the annular medial vertices is a cycle.

Consider the restriction map from proper 3-colourings of $\mathsf{M}(T)$ to colourings of this annular medial cycle $A(T)$. We claim that this map is injective. Let x be a non-annular medial vertex. Then x corresponds to an edge of B_{out} or B_{in}: chords of O are not incident to annular tread faces, and hence do not contribute vertices of $\mathsf{M}(T)$. This boundary edge is incident to a unique annular face of the tread, and the other two edges of that face are annular edges. Therefore x is adjacent in $\mathsf{M}(T)$ to the two annular medial vertices corresponding to those two annular edges.

Those two annular medial vertices are adjacent to each other, because their annular edges are consecutive on the same triangular annular face. In any proper 3-colouring they therefore receive two distinct colours, and x is forced to receive the remaining third colour. Thus every non-annular medial vertex has its colour uniquely determined by the colouring of $A(T)$. Two colourings of $\mathsf{M}(T)$ with the same restriction to $A(T)$ are identical, so the restriction map is injective. The stated inequality follows. □

Definition 3.4 (Annular teeth)**.** By Theorem 3.3 the annular medial vertices induce the cycle $A(T)$ in $\mathsf{M}(T)$, the annular cycle, so the edges of $\mathsf{M}(T)$ joining two annular medial vertices are exactly the edges of $A(T)$. Each such edge lies in exactly one triangle (3-cycle) of $\mathsf{M}(T)$, and the third vertex of that triangle is necessarily non-annular, since $A(T)$ has no chords. We call this triangle an *annular tooth* and its non-annular vertex the *apex* of the tooth.

The cycle $A(T)$ separates the plane into two regions: the *outer region*, which contains the outer-boundary medial vertices, and the *inner region*, which contains the inner-boundary medial vertices. An annular tooth is an *up tooth* if its apex lies in the outer region, and a *down tooth* if its apex lies in the inner region.

Remark 3.5. The apexes of annular teeth satisfy two sharing bounds: no two up teeth share an apex, and at most two down teeth share an apex.

Remark 3.6. The number of up teeth in $\mathsf{M}(T)$ is at least three.

Definition 3.7 (Bites)**.** By Remark 3.5 an apex is shared by at most two down teeth. When two down teeth share an apex, the pair is called a *bite*, and their common apex is the *apex of the bite*. A down tooth that belongs to a bite is a *bite tooth*. We further require the two annular edges carrying the teeth of a bite to be *non-incident*: they share no annular vertex of $A(T)$. Equivalently, the two bite teeth meet only at their common apex.

Remark 3.8. Let $B(T)$ be the subgraph of $\mathsf{M}(T)$ consisting of $A(T)$ together with all bite apexes (equivalently, $A(T)$ together with all bite teeth), drawn as a plane graph. For every interior face of $B(T)$ that is not a bite tooth, the number of down teeth whose apex lies in the interior of that face is either 0 or at least 3. Such an apex is necessarily that of a singleton down tooth: every bite apex is a vertex of $B(T)$, so it lies on a face boundary rather than in any face interior.

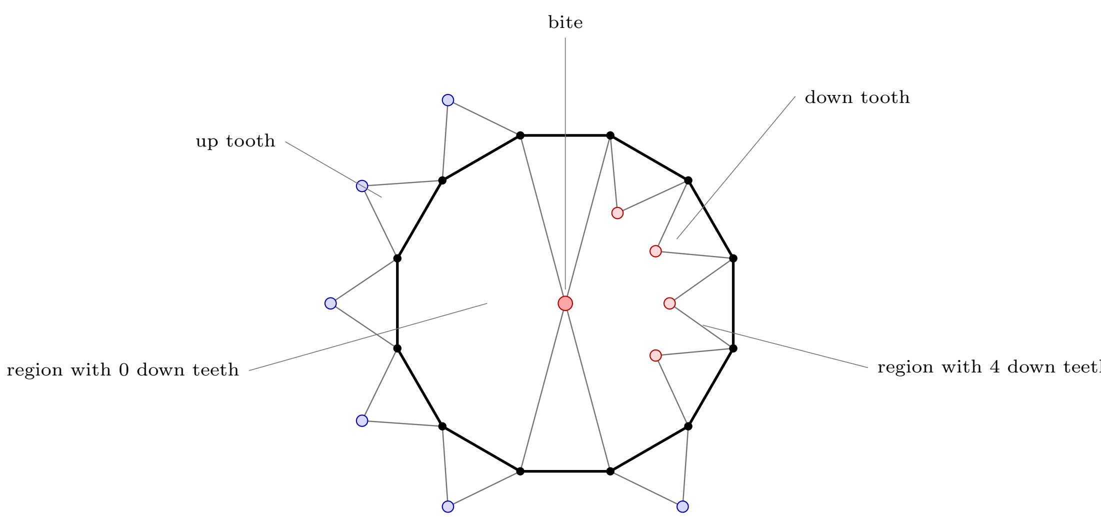

FIGURE 1. A full medial tire graph $\mathsf{M}(T)$ illustrating the tooth terminology. The thick cycle is the annular medial cycle $A(T)$, whose black vertices are the annular medial vertices. Each edge of $A(T)$ carries one tooth: up teeth (blue apexes, outer-boundary medial vertices) point into the outer region, and down teeth (red apexes, inner-boundary medial vertices) point into the inner region. The two down teeth meeting at the central shared apex (larger red vertex) form a bite; that shared apex splits the inner region into two faces, one with four down teeth on its boundary and one with none.

Definition 3.9 (Boundary medial vertices)**.** Let T be a tire tread and let Γ_T be the corresponding dual tire subgraph in G^*. A vertex $m_e \in V(\mathsf{M}(T))$ is an *outer boundary medial vertex* if the corresponding dual edge $e^* \in E(G^*)$ lies on the outer boundary of Γ_T. It is an *inner boundary medial vertex* if e^* lies on the inner boundary of Γ_T. We write

$$\partial_{\text{out}}\mathsf{M}(T) \quad \text{and} \quad \partial_{\text{in}}\mathsf{M}(T)$$

for the two boundary sets.

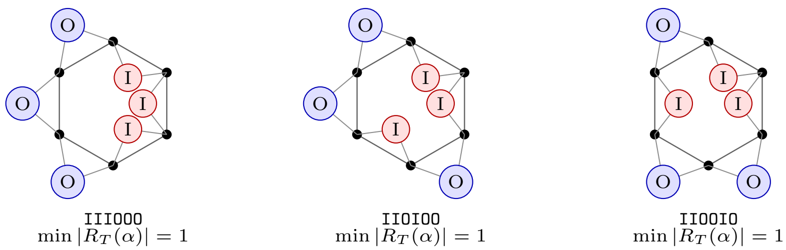

FIGURE 2. Three six-face full medial tire graphs found by the boundary-state restriction search. Black vertices are annular medial vertices; blue vertices are outer boundary medial vertices and red vertices are inner boundary medial vertices. The word below each diagram records the outer/inner type of the six annular faces in cyclic order. Boundary states are identified only up to colour permutation, not by rotation or reflection of the boundary order.

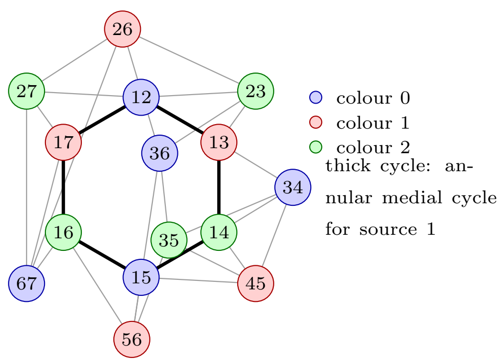

FIGURE 3. A proper vertex 3-colouring of the full medial graph of the first seven-vertex counterexample found by the experiment. The medial vertex labelled ij corresponds to the edge (i, j) of the triangulation. For the vertex-source decomposition at source 1, the highlighted annular medial cycle has colour counts $(2, 2, 2)$, so it is not coloured with two colours except at at most one vertex.

Definition 3.10 (Medial tire restriction relation)**.** Let $\mathrm{Col}_3(X)$ denote the set of proper vertex 3-colourings of the induced subgraph on a vertex set X. The *medial tire restriction relation* of T is

$$R_T \subseteq \mathrm{Col}_3(\partial_{\mathrm{out}}\mathsf{M}(T)) \times \mathrm{Col}_3(\partial_{\mathrm{in}}\mathsf{M}(T)),$$

where $(\alpha, \beta) \in R_T$ exactly when $\alpha \cup \beta$ extends to a proper vertex 3-colouring of $\mathsf{M}(T)$.

Remark 3.11. The definition deliberately records boundary colourings on medial vertices corresponding to boundary edges in the dual tire graph. Under Corollary 2.4, these are precisely edge-colouring states on the boundary edges through which a dual tire piece meets its parent and children.

4. Decomposition

Corollary 4.1 (Medial tire decomposition)**.** *Let G be a plane triangulation with level source S. The tire-tree decomposition $\mathcal{T}(G,S)$ of* [1] *induces a rooted decomposition of the full medial graph $M(G)$ into full medial tire graphs $\{\mathsf{M}(T) : T \in V(\mathcal{T}(G,S))\}$, glued along their boundary medial vertex sets.*

Proof. By the tire-tread partition theorem of [1], the bounded triangular faces of G are partitioned into nested tire treads, with intersections between parent and child treads occurring only along their level-cycle interface data. Every edge of G that is incident to a bounded face therefore belongs to the closure of at least one tire tread, and an edge lying in two closures lies on the interface between adjacent treads in the tire tree. Passing to $M(G)$ sends edges of G to medial vertices. Thus each tread determines the induced subgraph $\mathsf{M}(T)$ on its incident edge set, and overlaps between two such subgraphs are exactly the medial vertices corresponding to interface edges, namely the appropriate boundary medial vertex sets. □

Definition 4.2 (Compatible family of medial tire colourings)**.** A *compatible family of medial tire colourings* on $\mathcal{T}(G,S)$ is a choice, for each tread T, of a proper vertex 3-colouring φ_T of $\mathsf{M}(T)$ such that whenever T' is a child tread of T, the two colourings agree on $V(\mathsf{M}(T)) \cap V(\mathsf{M}(T'))$.

Proposition 4.3 (Gluing criterion)**.** *The full medial graph $M(G)$ has a proper vertex 3-colouring if and only if the tire tree $\mathcal{T}(G,S)$ admits a compatible family of medial tire colourings.*

Proof. A proper vertex 3-colouring of $M(G)$ restricts to a proper vertex 3-colouring of every induced subgraph $\mathsf{M}(T)$, and these restrictions agree on overlaps.

Conversely, suppose a compatible family is given. Define a colour on each vertex m_e of $M(G)$ by choosing any tread T with $m_e \in V(\mathsf{M}(T))$ and setting $\varphi(m_e) = \varphi_T(m_e)$. Compatibility makes this independent of the choice of T. Every medial edge of $M(G)$ is drawn in a corner of some bounded triangular face of G or along the outer boundary interface. The relevant incident primal edges lie together in the closure of a single tire tread or in a shared boundary interface, where properness is already enforced by one of the local colourings. Hence φ is a proper vertex 3-colouring of $M(G)$. □

5. A medial pigeonhole programme

The restriction relation R_T records exactly the local information needed to pass a medial 3-colouring through a tire. In a nested chain

$$T_0 \supset T_1 \supset \cdots \supset T_k,$$

the outer boundary state of T_{i+1} must match an inner boundary state allowed by R_{T_i}. Thus a proof of the Four Colour Theorem in this framework would follow from a structural reason that these restriction sets cannot remain mutually disjoint along every branch of the tire tree.

Definition 5.1 (Medial boundary state)**.** A *medial boundary state* on a boundary set $\partial\mathsf{M}(T)$ is a proper vertex 3-colouring of the subgraph induced by that boundary set, considered up to permutation of the three colours and the dihedral symmetries of the boundary walk when that boundary is a cycle.

Conjecture 5.2 (Medial chain-pigeonhole principle)**.** *There is a function $N(k)$ such that the following holds. Let $T_0 \supset T_1 \supset \cdots \supset T_{N(k)}$ be a nested chain of tire treads whose relevant boundary medial walks have length at most k. Then two adjacent restriction relations in the chain have compatible medial boundary states after colour permutation and boundary symmetry. Equivalently, the chain contains a local gluing step that cannot be obstructed by disjoint proper vertex 3-colouring restrictions.*

Conjecture 5.3 (Medial tire route to the Four Colour Theorem)**.** *For every plane triangulation G and every level source S, the restriction relations $\{R_T : T \in V(\mathcal{T}(G,S))\}$ admit a compatible selection of boundary states across the tire tree. Hence $M(G)$ is properly vertex 3-colourable, G^* is properly 3-edge-colourable, and G is properly 4-vertex-colourable.*

Remark 5.4. Conjecture 5.3 is equivalent in strength to the Four Colour Theorem when combined with Tait's correspondence. The point of the formulation is not to weaken the target theorem, but to move the obstruction into finite boundary-state restrictions carried by annular medial tire pieces.

5.1. **Kempe-cycle conservation across medial tires.** We now record an additional structure carried by proper 3-colourings of medial graphs. This structure will be useful for describing how colourings glue across level cycles.

Let G be a plane triangulation and let $M = M(G)$ be its medial graph. Let

$$\varphi : V(M) \to \{1,2,3\}$$

be a proper 3-colouring of M. For a two-element colour set $P = \{a,b\} \subseteq \{1,2,3\}$, let M_P denote the subgraph of M induced by the vertices of colours a and b.

Since M is 4-regular and φ is proper, every vertex of M_P has degree 2 in M_P. Hence every component of M_P is a cycle. We call these components the P-Kempe cycles of φ.

Lemma 5.5 (Kempe chains are cycles)**.** *Let G be a plane triangulation, let $M = M(G)$, and let φ be a proper 3-colouring of M. For each $P \in \{\{1,2\},\{2,3\},\{3,1\}\}$, every component of M_P is a cycle.*

Proof. Let $v \in V(M_P)$. In the medial graph M, the vertex v has degree 4. Since φ is a proper 3-colouring, none of the neighbours of v has colour $\varphi(v)$. Thus all four neighbours of v have one of the two colours different from $\varphi(v)$.

In the medial graph of a plane triangulation, the neighbours of a medial vertex occur in two opposite pairs corresponding to the two faces incident with the corresponding edge of G. Around each such triangular face, the three medial vertices receive all three colours. Consequently, at v there are exactly two neighbours of each colour different from $\varphi(v)$. It follows that, in the subgraph induced by any two colours P, every vertex has degree 2. Hence each component of M_P is a cycle. □

Let T be a medial tire region. We regard T as an annular transition region whose boundary consists of one outer level cycle and finitely many inner level cycles:

$$\partial T = C_0 \sqcup C_1 \sqcup \cdots \sqcup C_m.$$

Here C_0 is the outer level cycle of T, and the cycles $C_1, \ldots, C_m$ are the inner level cycles. Each inner level cycle C_i is also the outer level cycle of the corresponding child region in the tire tree.

The following lemma is the basic conservation principle.

Lemma 5.6 (Kempe-cycle conservation across level cycles)**.** *Let C be a level cycle of M separating a parent side from a child side. Let K be a P-Kempe cycle for some $P \in \{\{1,2\},\{2,3\},\{3,1\}\}$. Then K cannot enter the child side of C without also leaving it.*

Equivalently, the incidences of K with C are paired by the components of K lying on the child side of C, and also paired by the components of K lying on the parent side of C.

Proof. By the preceding lemma, K is a cycle. The level cycle C separates the sphere into two closed regions, which we call the parent side and the child side. Consider the intersection of K with one of these regions. Since K is a cycle, no component of this intersection can have exactly one boundary endpoint on C. Each component is either closed within the region, or is a path with two boundary endpoints on C. Thus every entrance through C is paired with an exit through C. □

We now use these Kempe cycles to single out the colourings of a full medial tire graph that respect the annular tooth structure.

Definition 5.7 (Kempe-balanced colouring)**.** Let φ be a proper 3-colouring of the full medial tire graph $\mathsf{M}(T)$. For a colour pair $P = \{a, b\}$, let $\mathsf{M}(T)_P$ be the subgraph induced by the vertices of colours a and b. Since $\mathsf{M}(T)$ need not be 4-regular, the components of $\mathsf{M}(T)_P$ are paths or cycles; we call them the *P-Kempe chains* of φ. Every vertex of colour a or b lies on exactly one P-Kempe chain.

A *valid face* is the outer face of $\mathsf{M}(T)$, or an interior face of $B(T)$ that is not a tooth—namely the root face or a bite inner-gap face of Remark 3.8. The *tooth apexes incident to* a valid face F are:

- the up-tooth apexes (Definition 3.4), when F is the outer face;
- the singleton down-tooth apexes whose annular edge lies on F, when F is interior—the apex on annular edge m being incident to the innermost bite (i, j) with $i < m < j$, or to the root face if there is none.

Bite apexes are never incident to a valid face in this sense.

For a colour pair $P = \{a, b\}$ write $\nu_P(F)$ for the number of tooth apexes incident to F that are coloured a or b—equivalently, that lie on a P-Kempe chain. The colouring φ is *Kempe-balanced* if $\nu_P(F)$ is even for every valid face F and every colour pair P.

Remark 5.8 (Necessity of Kempe-balance). A proper 3-colouring of $\mathsf{M}(T)$ can be part of a proper 3-colouring of the whole medial graph $M(G)$ only when it is Kempe-balanced: if φ is the restriction to $\mathsf{M}(T)$ of a proper 3-colouring of $M(G)$, then φ is Kempe-balanced. Equivalently, a colouring of $\mathsf{M}(T)$ that fails the parity condition at some valid face and colour pair cannot extend to a proper 3-colouring of $M(G)$. This is an instance of Kempe-cycle conservation (Lemma 5.6). The tooth apexes incident to a valid face are boundary medial vertices (Definition 3.9) lying on a single level cycle of the tire decomposition: the up-tooth apexes lie on the outer level cycle, and the singleton down-tooth apexes incident to an interior non-tooth face lie on the inner level cycle bounding that face. In the 4-regular graph $M(G)$ each P-Kempe chain of $\mathsf{M}(T)$ closes up into a P-Kempe cycle, which by Lemma 5.6 meets each level cycle in an even number of P-coloured incidences; for a given valid face these incidences are exactly its incident tooth apexes coloured a or b, whence $\nu_P(F)$ is even.

This argument is verified computationally. For bite-free pieces—capped triangulated annuli on annular cycles of length $6, 8, 10, 12$—every proper 3-colouring of $M(G)$ restricts to a Kempe-balanced colouring. The same holds for pieces carrying a bite, including the case where singleton down teeth lie in the bite's inner-gap face: there the inner level cycle splits into a child level cycle per gap, and conservation across each child cycle supplies the parity (in the checked example the three singleton down apexes of a bite gap are a rainbow in every restriction).

More generally, let T be a medial tire region with boundary

$$\partial T = C_0 \sqcup C_1 \sqcup \cdots \sqcup C_m.$$

For a P-Kempe cycle K, every component of $K \cap T$ is either a cycle contained in T, or a path with two endpoints on ∂T. Thus the P-Kempe arcs inside T define a pairing of the P-coloured boundary incidences of

$$C_0 \sqcup C_1 \sqcup \cdots \sqcup C_m.$$

This motivates the following refinement of boundary states.

Definition 5.9 (Kempe-enhanced boundary state)**.** Let T be a medial tire region with outer level cycle C_0 and inner level cycles $C_1, \dots, C_m$. Let

$$\mathcal{C}(T) = C_0 \sqcup C_1 \sqcup \cdots \sqcup C_m.$$

A *Kempe-enhanced boundary state* on T consists of the following data:

(1) a boundary colouring

$$\alpha : V(\mathcal{C}(T)) \to \{1, 2, 3\};$$

(2) for each colour pair

$$P \in \{\{1, 2\}, \{2, 3\}, \{3, 1\}\},$$

a pairing π_P of the P-coloured boundary incidences of $\mathcal{C}(T)$ induced by the P-Kempe arcs lying inside T.

We write such a state as

$$\kappa = (\alpha, \pi_{12}, \pi_{23}, \pi_{31}).$$

Given a proper 3-colouring φ of the medial tire graph $M(T)$, the restriction of φ to the boundary level cycles gives the boundary colouring α, while the two-colour Kempe arcs inside T give the pairings $\pi_{12}, \pi_{23}, \pi_{31}$. Thus φ determines a Kempe-enhanced boundary state, denoted

$$\kappa_T(\varphi).$$

Definition 5.10 (Kempe-enhanced restriction relation)**.** The *Kempe-enhanced restriction relation* of T is

$$\mathcal{K}_T = \{\kappa_T(\varphi) : \varphi \text{ is a proper 3-colouring of } M(T)\}.$$

This refines the ordinary boundary-colouring relation by recording not only which boundary colourings extend across T, but also how the two-colour Kempe cycles are routed through the annular tire region.

The annular structure of a tire is useful in two distinct ways. First, it gives a bounded transition region between level cycles: the colouring of the annular medial cycle controls, and in many cases determines, the colouring of the remaining medial tire vertices. Thus the number of possible transition states is bounded in terms of the annular structure, rather than the total size of the subtree below the tire. Second, it describes how the outer level cycle and the inner level cycles are related by Kempe arcs. The level cycles are the gluing interfaces, while the annular tire is the transition operator between them.

Definition 5.11 (Kempe-compatible gluing)**.** Let T be a medial tire region and let U be a child region glued to T along a common level cycle C. Thus C is an inner level cycle of T and the outer level cycle of U.

Let

$$\kappa_T = (\alpha_T, \pi_{12}^T, \pi_{23}^T, \pi_{31}^T) \in \mathcal{K}_T$$

and

$$\kappa_U = (\alpha_U, \pi_{12}^U, \pi_{23}^U, \pi_{31}^U) \in \mathcal{K}_U.$$

We say that κ_T and κ_U are *Kempe-compatible along* C if:

(1) the boundary colourings agree on C:

$$\alpha_T|_{V(C)} = \alpha_U|_{V(C)};$$

(2) for each colour pair

$$P \in \{\{1,2\},\{2,3\},\{3,1\}\},$$

the pairings π_P^T and π_P^U compose along the P-coloured incidences of C without producing an unpaired endpoint.

When these conditions hold, the composed pairings determine a Kempe-enhanced boundary state on the exposed boundary of $T \cup_C U$.

In these terms, gluing local colourings is not merely a matter of matching boundary colours. The colourings must also route their two-colour Kempe arcs compatibly across every shared level cycle. The ordinary restriction relation records whether a boundary colouring can be extended locally; the Kempe-enhanced relation additionally records the conservation of Kempe-cycle flow through the annular transition region.

For a tire with one outer level cycle and several inner level cycles,

$$\partial T = C_0 \sqcup C_1 \sqcup \cdots \sqcup C_m,$$

the parent tire may correlate the boundary states on the different inner cycles. The Kempe-enhanced relation records this correlation as a system of pairings among the P-coloured incidences of all boundary level cycles simultaneously. Thus one should view a medial tire as a multi-output transition operator

$$\mathcal{K}_T : C_0 \rightsquigarrow (C_1, \ldots, C_m),$$

rather than as an independent collection of binary transitions.

The guiding principle is therefore:

> Level cycles are the interfaces used for gluing, while annular tire regions are the bounded transition regions that route Kempe cycles between those interfaces.

References

[1] E. Bauerfeld, *Nested Tire Decompositions of Plane Triangulations*, manuscript (math-research repository), 2026.
[2] P. G. Tait, *Remarks on the colourings of maps*, Proceedings of the Royal Society of Edinburgh **10** (1880), 729–729.