

HEAWOOD RESTRICTIONS ON NESTED TIRE GRAPH DUALS

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ABSTRACT.

1. INTRODUCTION

A classical theorem of Tait recasts the Four Colour Theorem in dual, edge-colouring terms: a plane triangulation G is properly 4-vertex-colourable if and only if its dual cubic graph G' is properly 3-edge-colourable. Thus a minimal counterexample to the Four Colour Theorem – a smallest triangulation admitting no proper 4-colouring – corresponds to a smallest cubic plane graph admitting no proper 3-edge-colouring.

This paper continues the series studying that structure through the lens of *nested level duals*. The foundational vocabulary — level sources, levels, the inner planar dual G' and its dual depth, and tire graphs — is developed in the companion paper [3]; we refer to that paper for those definitions and rely on them throughout. In particular we use, without restating, the notions of:

- *level source* S and G -vertex levels $\ell_G(v)$;
- the inner planar dual G' ([3, Definition 1.3]);
- *dual depth* $\delta_G(d_f)$ ([3, Definition 1.4]);
- *tire graph* $T = (B_{\text{out}}, O, E_{\text{ann}})$ with outer/inner boundaries and annular edges ([3, Definition 1.5]);
- the *tire-component lemma* ([3, Lemma 1.8]); and
- the *tire-tread partition theorem* ([3, Theorem 1.9]).

Throughout, $G = (V, E)$ is a plane maximal planar graph (a triangulation) with a fixed planar embedding Π_G . We write $|V| = n$, so $|E| = 3n - 6$ and G has $2n - 4$ triangular faces.

2. HEAWOOD RESTRICTIONS ON THE TIRE DUAL

REFERENCES

- [1] P. J. Heawood, *On the four-colour map theorem*, Quart. J. Pure Appl. Math. **29** (1898), 270–285.
- [2] E. Bauerfeld, *Plane Depth*, manuscript (math-research repository), 2026.
- [3] E. Bauerfeld, *Nested Tire Decompositions of Plane Triangulations*, manuscript (math-research repository), 2026.
- [4] E. Bauerfeld, *Coloring Nested Tire Dual Graphs*, manuscript (math-research repository), 2026.

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