DUAL DECOMPOSITION OF MINIMAL COUNTEREXAMPLES

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Abstract.

1. The minimal counterexample

Throughout, a *triangulation* is a simple plane graph, with a fixed embedding, in which every face — including the outer face — is bounded by a triangle. We first reduce to triangulations, then record the degree properties a smallest counterexample must have.

Lemma 1.1 (Reduction to triangulations)**.** *If every triangulation is properly* 4*-vertex-colourable, then so is every plane graph.*

Proof. Let H be a plane graph. Add edges to H, maintaining planarity, until no further edge can be added; the result is a triangulation H^+ on the same vertex set with $E(H) \subseteq E(H^+)$. A proper 4-colouring of H^+ restricts to a proper 4-colouring of H, since every edge of H is an edge of H^+. □

By Lemma 1.1, if the Four Colour Theorem fails then it fails for some triangulation. We may therefore make the following assumption.

Definition 1.2 (Minimal counterexample)**.** Let G be a triangulation on the fewest vertices that admits no proper 4-vertex-colouring. We call G a *minimal counterexample.* By minimality, every triangulation on fewer than $|V(G)|$ vertices is properly 4-colourable.

Remark 1.3. Since every triangulation on at most four vertices is properly 4-colourable (the largest being K_4), a minimal counterexample has $|V(G)| \geq 5$; the degree bound below sharpens this to $|V(G)| \geq 12$.

Lemma 1.4 (Minimum degree)**.** *A minimal counterexample G has minimum degree $\delta(G) \geq 5$.*

Proof. Suppose some vertex v has $\deg(v) = d \leq 4$.

If $d \leq 3$, let $G' = G - v$. Then G' is a plane graph on fewer vertices, so by Definition 1.2 and Lemma 1.1 it has a proper 4-colouring. The at most three neighbours of v use at most three colours, so a fourth colour is free for v, extending the colouring to G — a contradiction.

If $d = 4$, again 4-colour $G - v$. If the four neighbours of v use at most three colours we extend as before, so assume they receive all four colours; let v_1, v_2, v_3, v_4 be the neighbours in cyclic order around v, coloured $1, 2, 3, 4$. Consider the subgraph

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induced by the colour classes 1 and 3, and let K be its connected component containing v_1. If $v_3 \notin K$, swap colours 1 and 3 on K; now no neighbour of v is coloured 1, freeing it for v. If $v_3 \in K$, then a 1–3 Kempe chain joins v_1 to v_3, and this chain together with v encloses exactly one of v_2, v_4; hence the 2–4 component containing v_2 cannot also reach v_4, and swapping colours 2 and 4 on it frees colour 2 for v. Either way the colouring extends to G, a contradiction.

Hence $\delta(G) \geq 5$. □

2. The reduced dual

Write G' for the dual of G: since G is a triangulation, G' is a cubic plane graph in which each vertex of G corresponds to a face of G', each face of G to a vertex of G', and each edge to a dual edge. A vertex of G of degree k corresponds to a k-gonal face of G'.

By Lemma 1.4, $\delta(G) \geq 5$, and Euler's formula gives $\sum_{u \in V(G)}(6 - \deg u) = 12$, so G has a vertex of degree exactly 5 (indeed at least twelve). Fix such a vertex v. Its dual face F_v is a pentagon, bounded by the five dual vertices corresponding to the five faces of G incident to v.

Definition 2.1 (Reduced dual)**.** Let v be a degree-5 vertex of G with pentagonal dual face F_v, and fix an index $i \in \{0, 1, 2, 3, 4\}$. The *reduced dual* $\widehat{G}'_{v,i}$ is the plane graph obtained from G' as follows.

(1) Delete the five dual vertices on the boundary of F_v, together with all edges incident to them. Each deleted vertex is cubic, with two edges on ∂F_v and one edge leaving F_v; deleting the five boundary vertices therefore removes the five external edges as well, dropping their five outer endpoints from degree 3 to degree 2. These five degree-2 vertices lie on the boundary of a single face F of the resulting graph.
(2) List the five degree-2 vertices in clockwise order around F as $A = (A_0, A_1, A_2, A_3, A_4)$.
(3) Add a new vertex v_n and join it to A_i, A_{i+1}, and A_{i+2} (indices mod 5) by three new edges.
(4) Add a new edge between A_{i+3} and A_{i+4} (indices mod 5).

Remark 2.2. Steps (3) and (4) restore cubicity: A_i, A_{i+1}, A_{i+2} each gain one edge to v_n and A_{i+3}, A_{i+4} each gain the new edge, so all five return to degree 3, and v_n has degree 3. Since $A_i, \ldots, A_{i+2}$ and A_{i+3}, A_{i+4} are each consecutive along ∂F, the new vertex and edge can be drawn inside F without crossings, so $\widehat{G}'_{v,i}$ is again a cubic plane graph. The construction depends on the choice of i up to the rotational symmetry of A.

Definition 2.3 (Edges of the reduced dual)**.** The four edges added in steps (3) and (4) of Definition 2.1 are named as follows. The chord $A_{i+3}A_{i+4}$ is the *merged edge*; the edge $A_{i+1}v_n$ is the *spike edge*; the edge $A_i v_n$ is the *side*-0 *edge*; and the edge $A_{i+2}v_n$ is the *side*-1 *edge*. In the $i = 0$ case of Figure 1 these are $\{A_3, A_4\}$, $\{A_1, v_n\}$, $\{A_0, v_n\}$, and $\{A_2, v_n\}$ respectively.

We will use the following structural fact about proper 3-edge-colourings near a pentagonal face of a cubic plane graph; it is stated for a generic such graph H, not specifically for the reduced dual.

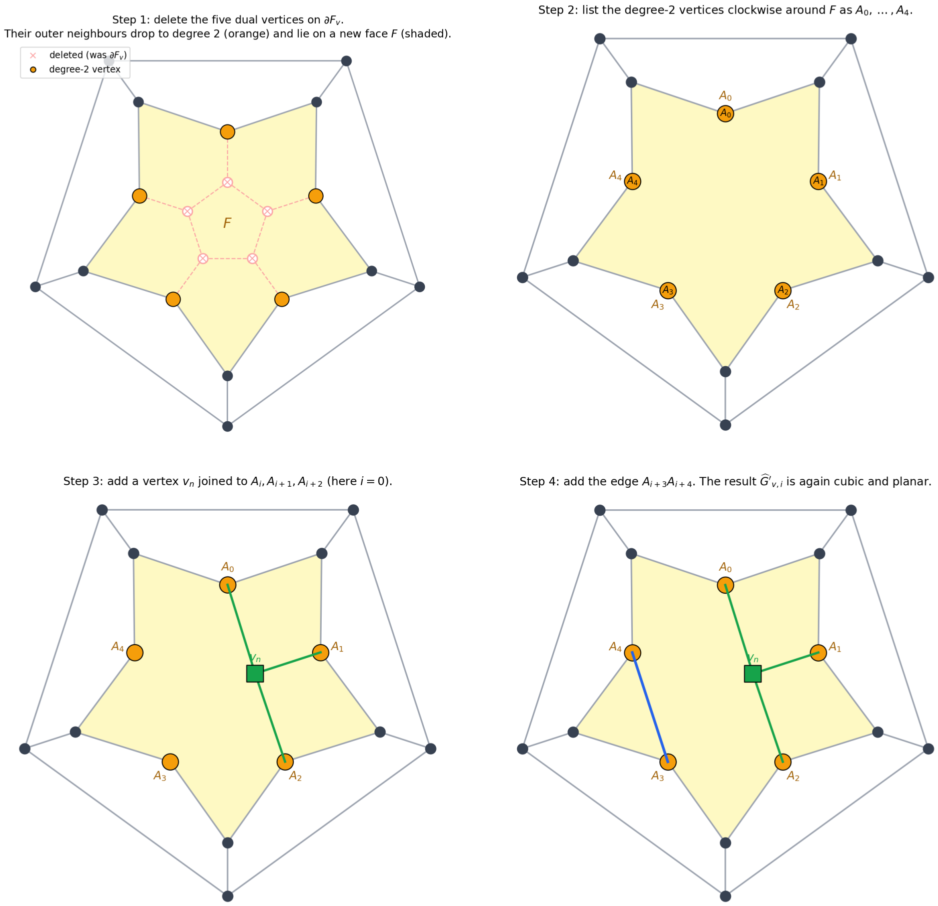

FIGURE 1. The four steps of Definition 2.1, illustrated on G' = the dodecahedron (dual of the icosahedron) with F_v the inner pentagon and $i = 0$. Top left: delete the five boundary vertices of F_v, leaving five degree-2 vertices on a new face F. Top right: order them clockwise as $A_0, \ldots, A_4$. Bottom left: add v_n joined to A_0, A_1, A_2. Bottom right: add the chord A_3A_4, giving the cubic plane graph $\widehat{G}'_{v,0}$.

Lemma 2.4 (Pentagonal externals)**.** *Let H be a cubic plane graph and F a pentagonal face of H, with ∂F traversed clockwise as u_0, u_1, u_2, u_3, u_4. For each i let f_i be the unique edge of H incident to u_i that does not lie on ∂F. An assignment φ of colours from $\{1,2,3\}$ to the ten edges incident to $\{u_0, \ldots, u_4\}$ is proper at every u_i if and only if there is some index j such that*

$$\varphi(f_j) = \varphi(f_{j+1}) = \varphi(f_{j+2}) \quad \textit{and} \quad \{\varphi(f_{j+3}), \varphi(f_{j+4})\} = \{1,2,3\} \setminus \{\varphi(f_j)\},$$

indices mod 5*.*

Proof. Write $e_i = u_iu_{i+1}$ for the boundary edges of ∂F (indices mod 5). A colouring φ is proper at every u_i if and only if at each u_i the three incident edges e_{i-1}, e_i, f_i

receive three distinct colours; whenever this holds, $\varphi(f_i)$ is forced to be the unique colour in $\{1,2,3\} \setminus \{\varphi(e_{i-1}), \varphi(e_i)\}$, and φ restricts to a proper 3-edge-colouring of the cycle ∂F.

(⇒) The line graph of ∂F is C_5, whose maximum independent set has size 2, so no colour appears more than twice on ∂F; and since ∂F is an odd cycle, all three colours appear. The colour multiset on $(\varphi(e_0), \ldots, \varphi(e_4))$ is therefore $(2,2,1)$, with the singleton at a unique position. Cyclically shifting indices we may place this position at 0; let c be the singleton colour. The remaining four edges form the path $e_1e_2e_3e_4$, which by propriety alternates between the other two colours, so for some labelling $\{a,b,c\} = \{1,2,3\}$,

$$(\varphi(e_0), \varphi(e_1), \varphi(e_2), \varphi(e_3), \varphi(e_4)) = (c, a, b, a, b).$$

Reading off the forced values of $\varphi(f_i)$,

$$\varphi(f_0) = a, \quad \varphi(f_1) = b, \quad \varphi(f_2) = \varphi(f_3) = \varphi(f_4) = c,$$

which is the lemma's pattern at $j = 2$ (the cyclic shift maps this back to the corresponding j in the original indexing). This case is the unique proper 3-edge-colouring of ∂F up to cyclic shift and permutation of $\{1,2,3\}$ (since $5 \cdot 3! = 30 = P(C_5, 3)$, the chromatic polynomial of C_5 at 3), so it exhausts every proper φ.

(⇐) The lemma's hypothesis is invariant under cyclic shifts of indices and under permutations of $\{1,2,3\}$, so we may assume $j = 2$, $\varphi(f_2) = \varphi(f_3) = \varphi(f_4) = c$, $\varphi(f_0) = a$, and $\varphi(f_1) = b$, with $\{a,b,c\} = \{1,2,3\}$. Propriety at u_i and u_{i+1} requires $\varphi(e_i) \notin \{\varphi(f_i), \varphi(f_{i+1})\}$, which gives

$$\varphi(e_0) = c, \quad \varphi(e_1) = a, \quad \varphi(e_2) \in \{a,b\}, \quad \varphi(e_3) \in \{a,b\}, \quad \varphi(e_4) = b.$$

The remaining propriety condition $\varphi(e_{i-1}) \neq \varphi(e_i)$ holds automatically at u_0, u_1, u_4, forces $\varphi(e_2) = b$ at u_2, and then forces $\varphi(e_3) = a$ at u_3. The resulting triples $(\varphi(e_{i-1}), \varphi(e_i), \varphi(f_i))$ at u_0, u_1, u_2, u_3, u_4 are

$$(b,c,a), \quad (c,a,b), \quad (a,b,c), \quad (b,a,c), \quad (a,b,c),$$

each a permutation of $\{1,2,3\}$, so φ is proper at every u_i. □

Remark 2.5. The two-element condition $\{\varphi(f_{j+3}), \varphi(f_{j+4})\} = \{1,2,3\} \setminus \{\varphi(f_j)\}$ cannot be dropped: a 3-colouring satisfying $\varphi(f_j) = \varphi(f_{j+1}) = \varphi(f_{j+2})$ alone need not extend, e.g. $(1,1,1,1,2)$.

Since $\widehat{G}'_{v,i}$ is the dual of a triangulation on fewer vertices than G, it is 3-edge-colourable by the minimality of G. The following lemma constrains every such colouring.

Lemma 2.6. *Let G be a minimal counterexample, and let $\widehat{G}'_{v,i}$ be a reduced dual of its dual G'. Then in every proper 3-edge-colouring of $\widehat{G}'_{v,i}$, the merged edge and the spike edge receive the same colour.*

Proof. After cyclically relabelling, assume $i = 0$. Suppose for contradiction that φ is a proper 3-edge-colouring of $\widehat{G}'_{v,0}$ in which the merged edge $\{A_3, A_4\}$ and the spike edge $\{A_1, v_n\}$ receive different colours (Figure 2, top), and write

$$X = \varphi(\{A_0, v_n\}), \quad Y = \varphi(\{A_1, v_n\}), \quad Z = \varphi(\{A_2, v_n\}), \quad W = \varphi(\{A_3, A_4\}).$$

Propriety of φ at v_n forces $\{X, Y, Z\} = \{1,2,3\}$, and the assumption $W \neq Y$ leaves $W \in \{X, Z\}$.

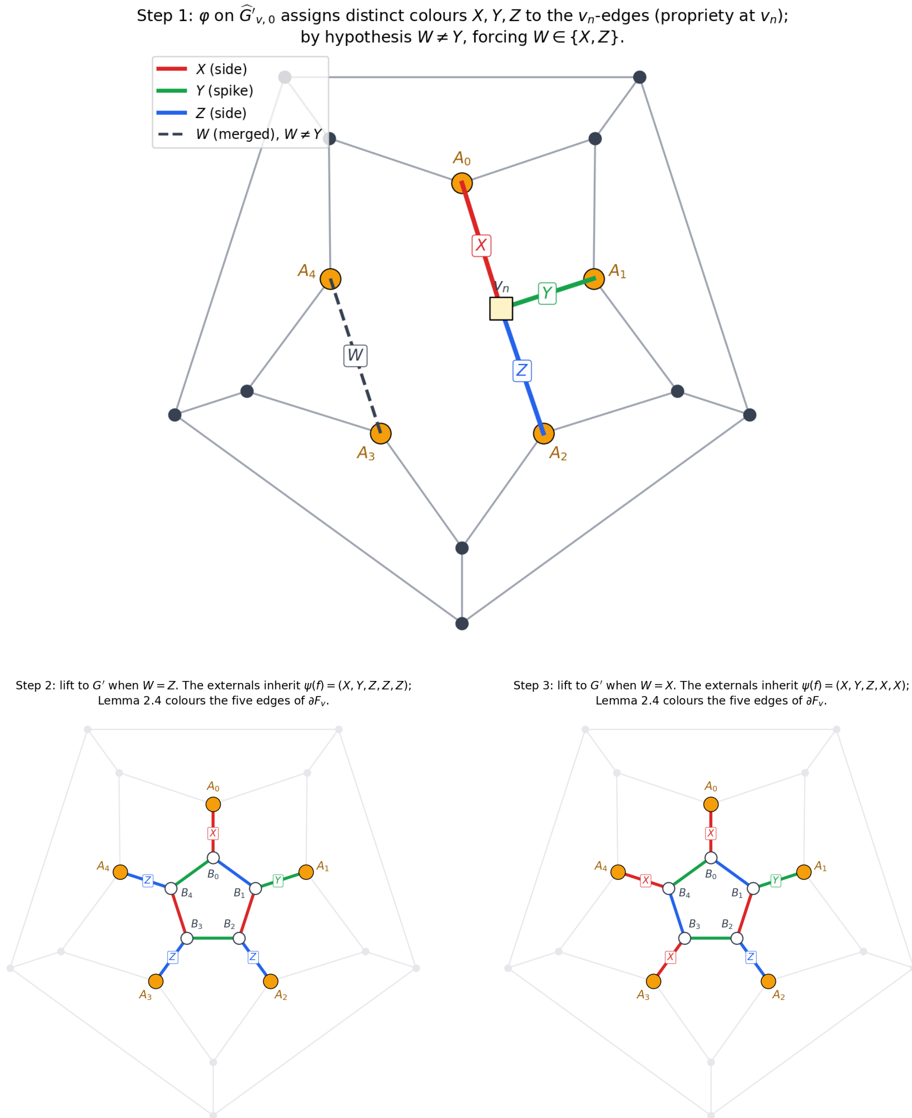

FIGURE 2. The proof of Lemma 2.6, illustrated for $i = 0$ on $G' =$ the dodecahedron. Top: under the assumption $W \neq Y$, propriety at v_n forces $W \in \{X, Z\}$. Bottom: in either case the lift to G' has externals satisfying the hypothesis of Lemma 2.4, which colours ∂F_v to extend ψ to a proper 3-edge-colouring of G'.

We lift φ to a colouring ψ of $E(G')$ as follows. Let $B_0, \ldots, B_4$ be the boundary vertices of ∂F_v in G', indexed so that $f_k = B_k A_k$. On every edge of G' that survived the reduction, set $\psi = \varphi$. At each A_k the two surviving edges retain their φ-colours, so the remaining edge at A_k — in G' this is the external f_k; in $\widehat{G}'_{v,0}$ this is a v_n-edge ($k \in \{0, 1, 2\}$) or the chord ($k \in \{3, 4\}$) — is forced to take the third colour at A_k. Since the two-surviving-edge colours at A_k agree in G' and $\widehat{G}'_{v,0}$, the third colour does too, giving

$$\psi(f_0) = X, \quad \psi(f_1) = Y, \quad \psi(f_2) = Z, \quad \psi(f_3) = \psi(f_4) = W$$

(the last two equalities holding because the chord is a single edge contributing its colour at each of A_3 and A_4).

It remains to assign colours to the five boundary edges B_kB_{k+1} of ∂F_v. Apply Lemma 2.4 to G' at the face F_v with the B_k's as its boundary vertices and the same indexing. The external vector $(\psi(f_0), \ldots, \psi(f_4)) = (X, Y, Z, W, W)$ falls into one of two cases (Figure 2, bottom):

- if $W = Z$, it is (X, Y, Z, Z, Z): three consecutive Z's at positions $2, 3, 4$, with $\{X, Y\} = \{1, 2, 3\} \setminus \{Z\}$;
- if $W = X$, it is (X, Y, Z, X, X): three consecutive X's at positions $3, 4, 0$, with $\{Y, Z\} = \{1, 2, 3\} \setminus \{X\}$.

Each case satisfies the hypothesis of Lemma 2.4; its ($\Leftarrow$) direction therefore assigns colours to the boundary edges B_kB_{k+1} that make ψ proper at every B_k.

The resulting ψ is a proper 3-edge-colouring of G': proper at every B_k by the lemma, at every A_k by the construction, and at every other vertex because such a vertex has the same neighbourhood in G' as in $\widehat{G}'_{v,0}$ with the same incident-edge colours. By Tait's theorem, G' is 3-edge-colourable iff G is 4-vertex-colourable, contradicting that G is a counterexample. The assumption $W \neq Y$ is therefore false. □

For a pair of colours $\{a, b\} \subseteq \{1, 2, 3\}$, the subgraph of $\widehat{G}'_{v,i}$ on the edges coloured a or b is 2-regular (since at each vertex exactly one of the three incident edges is excluded), and hence a disjoint union of cycles. We call each such cycle a $\{a, b\}$-*Kempe cycle*, and reserve the notation for the specific cycle containing a given edge when the context makes it clear. Swapping the two colours on a single Kempe cycle yields another proper 3-edge-colouring of the same graph.

Lemma 2.7 (Kempe cycles through the spike)**.** *Let G be a minimal counterexample, fix a reduced dual $\widehat{G}'_{v,i}$ of G', and let φ be a proper 3-edge-colouring of $\widehat{G}'_{v,i}$. Write c for the common colour assigned by φ to the spike and the merged edge (Lemma 2.6), and c_0, c_1 for the colours of the side-0 and side-1 edges respectively, so $\{c, c_0, c_1\} = \{1, 2, 3\}$. Then*

1. *the $\{c, c_0\}$-Kempe cycle through the spike edge contains both the side-0 edge and the merged edge;*
2. *the $\{c, c_1\}$-Kempe cycle through the spike edge contains both the side-1 edge and the merged edge.*

Proof. We prove (1); (2) is the same argument with c_1 and the side-1 edge in place of c_0 and the side-0 edge.

The spike edge $\{A_{i+1}, v_n\}$ and the side-0 edge $\{A_i, v_n\}$ share the vertex v_n and receive the two colours c, c_0, so they both lie on the $\{c, c_0\}$-Kempe cycle through v_n. Suppose for contradiction that the merged edge lies on a different $\{c, c_0\}$-Kempe cycle K (it lies on *some* such cycle, since it has colour c). Let φ' be obtained from φ by swapping the colours c and c_0 along K alone: this is a Kempe swap, so φ' is again a proper 3-edge-colouring of $\widehat{G}'_{v,i}$. Under φ' the spike edge — which is not on K — still has colour c, but the merged edge — which is on K — now has colour c_0.

Hence in φ' the spike and the merged edge receive distinct colours, contradicting Lemma 2.6 applied to φ'. □

3. An iterated reduction

The reduced-dual construction in Definition 2.1 can be iterated: starting from a proper 3-edge-colouring φ_1 of a reduced dual $\widehat{G}'_{v,i}$, we apply the construction again to that graph at a pentagonal face whose ten incident edges avoid the four named edges from the first reduction, extending φ_1 across the new reduction. The protected edges accumulate into a set E that grows by four per iteration, and the process terminates when E has blocked every pentagonal face.

Algorithm 3.1 (Iterated reduction with protected edges)**.** Let G be a triangulation we assume to be a minimal counterexample to the Four Colour Theorem. The algorithm produces a sequence $H_1, H_2, \ldots$ of cubic plane graphs, proper 3-edge-colourings φ_t of H_t, and a growing set E of protected edges.

(0) Form $G' := \mathrm{dual}(G)$, a cubic plane graph.
(1) Choose a degree-5 vertex v of G (equivalently a pentagonal face F_v of G') and an index $i_1 \in \{0, \ldots, 4\}$. Apply Definition 2.1 to form $H_1 := \widehat{G'}_{v,i_1}$, and fix any proper 3-edge-colouring φ_1 of H_1 (one exists by the minimality of G).
(2) Initialise $E := \{\mathrm{spike}, \mathrm{side\text{-}0}, \mathrm{side\text{-}1}, \mathrm{merged}\}$, the four named edges of the reduction in (1).
(3) (Iterate.) At step $t \geq 2$, given H_{t-1}, φ_{t-1}, and $E \subseteq E(H_{t-1})$:
   (a) Find a pentagonal face F of H_{t-1} whose ten incident edges — the five boundary edges of ∂F and the five external edges at ∂F — are all outside E. If no such F exists, terminate.
   (b) By Lemma 2.4 applied to H_{t-1} at F under φ_{t-1}, the external vector has shape (a, b, c, c, c) up to cyclic rotation. Choose an index i_t for which $\varphi_{t-1}(f_{i_t+3}) = \varphi_{t-1}(f_{i_t+4})$ and $\varphi_{t-1}(f_{i_t}), \varphi_{t-1}(f_{i_t+1}), \varphi_{t-1}(f_{i_t+2})$ are three distinct colours.
   (c) Apply Definition 2.1 to H_{t-1} at (F, i_t) to form H_t.
   (d) Extend φ_{t-1} to a proper 3-edge-colouring φ_t of H_t: every surviving edge keeps its φ_{t-1}-colour, and each new edge takes the unique colour completing the palette at its endpoint (consistent across both endpoints of the chord by the choice of i_t).
   (e) Add the four named edges of the step-t reduction to E.
(4) Repeat (3) until termination.

Remark 3.2. At each iteration, $|V(H_t)| = |V(H_{t-1})| - 4$ and $|E(H_t)| = |E(H_{t-1})| - 6$, so H_t shrinks at a fixed rate; the protected set $|E|$ grows by exactly four; and every protected edge survives all subsequent reductions. Since the graph is finite, termination is guaranteed. By Lemma 2.4, step (b) never fails: some valid i_t always exists for any pentagonal face under any proper colouring. Termination is therefore combinatorial: it occurs precisely when E touches every pentagonal face of H_{t-1}.

Remark 3.3. Lemma 2.6 applies only at $t = 1$, when H_1 is a reduced dual of G'. For $t \geq 2$, H_t is a reduced dual of H_{t-1} rather than of G', and H_{t-1} is itself 3-edge-colourable, so the non-3-edge-colourability argument that drives Lemma 2.6 does not carry over. Whether the constraints accumulated in E propagate any further structure to φ_t for $t \geq 2$ is left open.

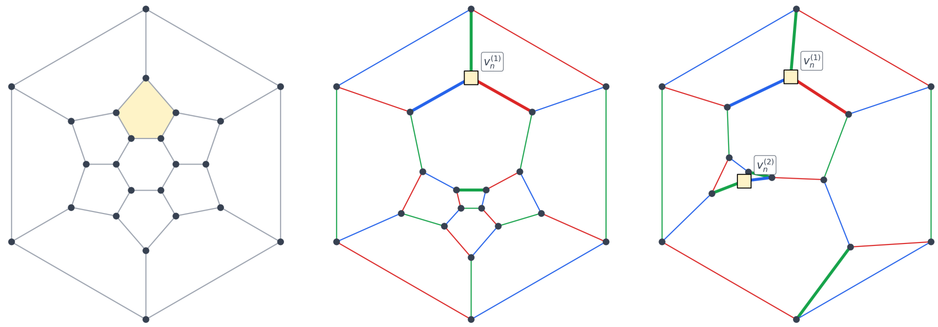

FIGURE 3. Algorithm 3.1 on $G' = \text{dual}(G)$, where G is the first min-degree-5 plantri triangulation on 14 vertices and φ_1 is a specific proper 3-edge-colouring of H_1 that satisfies both the chord-apex condition (Lemma 2.6) and the Kempe-cycle condition (Lemma 2.7), found by `experiments/search_kempe_property.py`. *Left:* G' (24 vertices, 36 edges) with the chosen pentagonal face shaded. *Centre:* H_1 (20 vertices, 30 edges) after step (1) with $i_1 = 1$, 3-edge-coloured by φ_1; the four edges around $v_n^{(1)}$ in E are drawn thicker, and the spike and merged edges share the colour green. *Right:* H_2 (16 vertices, 24 edges) after step (3) with $i_t = 3$; eight edges are protected, and the algorithm terminates one step later (no remaining safe pentagonal face in H_2). The generating script is `experiments/draw_iterated_reduction_n14.py`; layouts are Tutte barycentric embeddings with the outer face picked to keep $v_n^{(1)}, v_n^{(2)}$ in the interior.

Lemma 3.4 (Exactly one matching pair in the algorithm's output)**.** *Let G be a minimal counterexample to the Four Colour Theorem, and let (H_{t^*}, φ_{t^*}) be the final graph-and-colouring produced by some terminating execution of Algorithm 3.1 on G, with named pairs $(\text{spike}_t, \text{merged}_t)$ for $t = 1, \dots, t^*$. Then there is exactly one t with $\varphi_{t^*}(\text{spike}_t) = \varphi_{t^*}(\text{merged}_t)$, and it is $t = 1$.*

Proof. The algorithm never re-colours an existing edge: at each iteration step (3d) every surviving edge keeps its φ_{t-1}-colour, and the four new edges receive fresh colours forced by propriety. Hence for every $1 \le k \le t \le t^*$,

$$\varphi_t(\text{spike}_k) = \varphi_k(\text{spike}_k), \qquad \varphi_t(\text{merged}_k) = \varphi_k(\text{merged}_k);$$

the colours of the step-k named edges, once written, are permanent. It suffices to compare $\varphi_k(\text{spike}_k)$ and $\varphi_k(\text{merged}_k)$ at the step where each pair is introduced.

Case $k = 1$. Since G is a minimal counterexample, H_1 is a reduced dual of G'. Lemma 2.6 applied to φ_1 gives $\varphi_1(\text{spike}_1) = \varphi_1(\text{merged}_1)$.

Case $k \ge 2$. At step k the algorithm picks an index i_k for which $f_{i_k+3} = f_{i_k+4}$ (chord consistency) and $f_{i_k}, f_{i_k+1}, f_{i_k+2}$ are pairwise distinct (propriety at the new v_n), where f is the external vector of the chosen pentagonal face of H_{k-1} under φ_{k-1}. Step (3d) then assigns

$$\varphi_k(\text{spike}_k) = f_{i_k+1}, \qquad \varphi_k(\text{merged}_k) = f_{i_k+3}.$$

By Lemma 2.4, f has the $(2,2,1)$ pattern: a block of three consecutive positions $\{p, p+1, p+2\}$ (mod 5) on which it is constantly some colour c, while the remaining two positions $\{p+3, p+4\}$ hold the two non-c colours, one each. The condition $f_{i_k+3} = f_{i_k+4}$ forces (i_k+3, i_k+4) to be either $(p, p+1)$ or $(p+1, p+2)$ — the two consecutive pairs inside the block — and correspondingly $i_k + 1 \in \{p+3, p+4\}$, *outside* the block. So f_{i_k+1} is not c, whereas $f_{i_k+3} = c$, and hence $\varphi_k(\text{spike}_k) \neq \varphi_k(\text{merged}_k)$.

Combining the two cases, exactly one $t \in \{1, \ldots, t^*\}$ — namely $t = 1$ — has $\varphi_{t^*}(\text{spike}_t) = \varphi_{t^*}(\text{merged}_t)$. □

Lemma 3.5 (All-distinct colouring exists on a 4-colourable graph)**.** *Let G be a 4-colourable maximal planar graph of minimum degree ≥ 5 (equivalently, a maximal planar graph that is* not *a minimal counterexample to the Four Colour Theorem). Then there is an execution of Algorithm 3.1 on G whose final colouring φ_{t^*} satisfies $\varphi_{t^*}(\text{spike}_t) \neq \varphi_{t^*}(\text{merged}_t)$ for every $t \in \{1, \ldots, t^*\}$. In particular, there exists a proper 3-edge-colouring of H_{t^*} under which every spike-merged pair has distinct colours.*

Proof. The argument mirrors Lemma 3.4, but extends a colouring *downward* from G' rather than carrying one forward from H_1.

Since G is 4-colourable, by Tait's theorem $G' = \text{dual}(G)$ admits a proper 3-edge-colouring ξ. Apply Lemma 2.4 to ξ at the pentagonal face F_v selected in step (1): the external vector $f = (f_0, \ldots, f_4)$ at F_v under ξ has the $(3,1,1)$ cyclic-consecutive shape, with a block of three consecutive positions $\{p, p+1, p+2\}$ (mod 5) holding a common colour c, and the remaining two positions $\{p+3, p+4\}$ holding the two non-c colours, one each. The algorithm's choice of i_1 forces $\{i_1+3, i_1+4\}$ inside the c-block (so the chord is consistently coloured) and the three positions $\{i_1, i_1+1, i_1+2\}$ pairwise distinct; in particular $i_1 + 1$ lies *outside* the c-block.

Choose φ_1 to be the proper 3-edge-colouring of H_1 that agrees with ξ on every surviving edge and assigns each new edge at A_j the unique third colour at A_j. Then $\varphi_1(\text{spike}_1) = f_{i_1+1}$, a value not equal to c, while $\varphi_1(\text{merged}_1) = f_{i_1+3} = c$, so $\varphi_1(\text{spike}_1) \neq \varphi_1(\text{merged}_1)$.

The same argument repeats at every step $k \geq 2$: the external vector at the chosen pentagonal face under φ_{k-1} has the $(3,1,1)$ cyclic-consecutive shape (Lemma 2.4), the algorithm's index choice i_k puts i_k+3, i_k+4 inside the colour block and i_k+1 outside, and step (3d) thus assigns $\varphi_k(\text{spike}_k) \neq \varphi_k(\text{merged}_k)$. The algorithm preserves these colours through every later step, so $\varphi_{t^*}(\text{spike}_t) \neq \varphi_{t^*}(\text{merged}_t)$ for every $t \in \{1, \ldots, t^*\}$. □

Conjecture 3.6. *Let G be a minimal counterexample to the Four Colour Theorem, and let $\widehat{G}'_{v,i}$ be a reduced dual of $G' = \text{dual}(G)$. Then for every proper 3-edge-colouring φ of $\widehat{G}'_{v,i}$ there exist a face F of $\widehat{G}'_{v,i}$ and two distinct edges $e_1, e_2 \in \partial F$, with neither e_1 nor e_2 equal to the merged edge, such that*

(1) $\varphi(e_1) = \varphi(e_2)$,
(2) *e_1, e_2, and the merged edge all lie on a common $\{a,b\}$-Kempe cycle of φ, and*
(3) *exactly one edge of ∂F lies between e_1 and e_2 along one of the two arcs of ∂F; equivalently, subdividing e_1 and e_2 by new vertices X_1, X_2 and joining them by a new edge X_1X_2 inside F creates a new face bounded by exactly*

4 edges (the new edge, the two subdivision halves adjacent to it, and the single ∂F-edge between e_1 and e_2).

Remark 3.7. The conjecture cannot be tested on actual minimal counterexamples (none exist by the Four Colour Theorem), but its conclusion is checkable on the structural surrogates: proper 3-edge-colourings of reduced duals that satisfy both the chord-apex condition (Lemma 2.6) and the Kempe-cycle conditions (Lemma 2.7), since a counterexample's reduced dual is forced to admit such colourings under any proper colouring. For every min-degree-5 triangulation G with $|V(G)| \leq 21$, every pentagonal face F of G', and every reduction index $i \in \{0, \dots, 4\}$, we enumerated all such colourings and tested the three clauses of Conjecture 3.6 (see `experiments/check_conj_face_kempe_scaled.py`); $n = 22$ ran past a 1800 s budget after 641,700 colourings (all pass), but did not finish the full set of 651 triangulations:

n	#tri	#col. tested	#sat.	status
12	1	0	—	vacuous (icosahedron)
13	0	—	—	no min-deg-5 tri
14	1	216	216	all pass
15	1	0	—	vacuous
16	3	864	864	all pass
17	4	4,650	4,650	all pass
18	12	8,070	8,070	all pass
19	23	21,138	21,138	all pass
20	73	107,874	107,874	all pass
21	192	392,370	392,370	all pass
22 (part.)	651	641,700	641,700	timeout
total ($n \leq 21$)	311	535,182	535,182	

The vacuous rows ($n = 12, 15$) are those where the relevant reduced duals admit no proper 3-edge-colouring satisfying chord-apex + both Kempe-cycle conditions, so the conjecture has no content there. On every (G, F, i, φ) with content, all three clauses of the conjecture hold simultaneously.

The next definition records a cubic-preserving analogue of edge contraction which turns out — under planar duality — to coincide with simple-graph contraction on the dual side. It will be the central tool in Section 4 below, where we formulate a sufficient condition for the Four Colour Theorem.

Definition 3.8 (Cubic-graph edge contraction)**.** Let H be a cubic plane graph and $e = uv$ an edge of H with $u \neq v$ and no edge of H parallel to e. The *cubic-graph edge contraction* of H along e is the graph H' obtained in two steps:

(1) *Delete* the edge e; the endpoints u and v each drop to degree 2.
(2) *Smooth* each of u and v: at u, replace u and its two remaining incident edges ua, ub by a single new edge ab; do the same at v. Both vertices u and v are removed, and two new edges are added in their place.

Provided the smoothings do not introduce a loop or parallel edge, H' is again a cubic plane graph, with $|V(H')| = |V(H)| - 2$ and $|E(H')| = |E(H)| - 3$.

Equivalently, H' is the planar dual of $\mathrm{dual}(H)/e^*$, where e^* is the edge of $\mathrm{dual}(H)$ crossing e and the contraction on the right-hand side is simple-graph contraction (loops removed, parallel edges absorbed). Under planar duality, contracting e^* in dual(H) merges the two triangular faces of dual(H) incident to e^*, and the parallel-edge cleanup corresponds exactly to the smoothing step on the primal side.

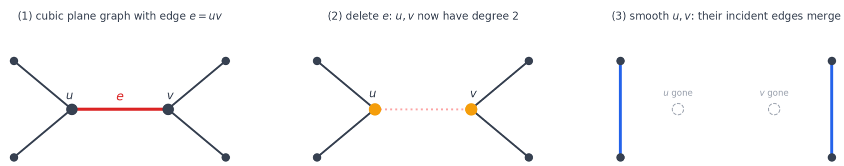

FIGURE 4. Cubic-graph edge contraction (Definition 3.8). Left: a fragment of a cubic plane graph with the contracted edge $e = uv$ highlighted in red. Middle: deleting e leaves u and v of degree 2. Right: smoothing u and v replaces each pair of incident edges by a single new edge, removing u, v and giving a cubic plane graph again.

Theorem 3.9 (Cubic contraction across a 4-face preserves 3-edge-colourability)**.** *Let H be a cubic plane graph with a proper 3-edge-colouring φ, let f be a face of H with $|\partial f| = 4$, and let e_0, e_1 be the two edges of ∂f sharing no endpoint (the opposite pair on the 4-cycle ∂f). If $\varphi(e_0) \neq \varphi(e_1)$ and the cubic-graph edge contraction of H along e_0 (Definition 3.8) is well-defined (no loops or parallel edges are created), then the contracted graph admits a proper 3-edge-colouring.*

Proof. Write ∂f as the 4-cycle $v_0v_1v_2v_3$ with $e_0 = v_0v_1$ and $e_1 = v_2v_3$ (so e_0, e_1 are opposite); the remaining two boundary edges of f are $e_2 := v_1v_2$ and $e_3 := v_3v_0$. Since H is cubic, each v_i has exactly one edge not on ∂f: write w_i for that edge and u_i for its other endpoint, so $w_i = v_iu_i$ with $u_i \notin \{v_0, v_1, v_2, v_3\}$, for each $i \in \{0, 1, 2, 3\}$. Put $a := \varphi(e_0)$, $b := \varphi(e_1)$, and let c be the third colour.

Forced colours on the face. Propriety at v_1 and v_2 forces $\varphi(e_2) \notin \{a, b\}$, so $\varphi(e_2) = c$; then $\varphi(w_1) = b$ and $\varphi(w_2) = a$. Symmetrically $\varphi(e_3) = c$, $\varphi(w_0) = b$, and $\varphi(w_3) = a$. In particular $\varphi(w_0) = \varphi(w_1) = b$.

Construction of φ'. Let H' denote the cubic-graph edge contraction of H along e_0; its new edges are $e_3' := v_3u_0$ (replacing e_3 and w_0 via the smoothing at v_0) and $e_2' := v_2u_1$ (replacing e_2 and w_1 via the smoothing at v_1). Define $\varphi' \colon E(H') \to \{1, 2, 3\}$ by

$$\varphi'(e) := \begin{cases} c & \text{if } e = e_1, \\ b & \text{if } e \in \{e_2', e_3'\}, \\ \varphi(e) & \text{otherwise.} \end{cases}$$

That is: give each smoothed-in edge the colour b (the colour of the two w_i it absorbs), recolour e_1 to c, and leave every other edge of H' with its φ-colour.

Propriety. Every vertex of H' other than v_2, v_3, u_0, u_1 has the same incident edges and the same φ'-colours as it did under φ, so propriety is inherited there. At the four affected vertices,

vertex	edges in H'	colours under φ'
v_2	$e_1,\ w_2,\ e_2'$	$c,\ a,\ b$
v_3	$e_1,\ w_3,\ e_3'$	$c,\ a,\ b$
u_0	$e_3',\ \alpha_0,\ \beta_0$	$b,\ a,\ c$
u_1	$e_2',\ \alpha_1,\ \beta_1$	$b,\ a,\ c$

where α_i, β_i are the two edges of H at u_i other than w_i, whose φ-colours are forced to $\{a, c\}$ by propriety at u_i (since $\varphi(w_i) = b$). Each row lists three distinct colours, so φ' is proper. □

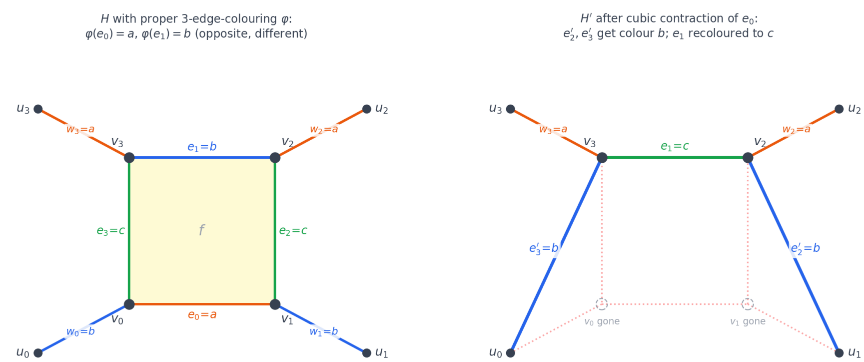

Figure 5. The recolouring used in the proof of Theorem 3.9. Left: the 4-face f of H under φ, with the forced colours $\varphi(e_0) = a$, $\varphi(e_1) = b$, $\varphi(e_2) = \varphi(e_3) = c$, $\varphi(w_0) = \varphi(w_1) = b$, and $\varphi(w_2) = \varphi(w_3) = a$. Right: the contracted graph H' under φ'. The smoothed-in edges e_2', e_3' inherit the colour b from w_0, w_1, and e_1 is recoloured from b to c; every edge outside the face neighbourhood keeps its φ-colour (dotted in red: the five edges of H removed by the contraction).

4. The Four Colour Theorem from a strengthened conjecture

The next conjecture strengthens Conjecture 3.6 by adding a clause that arranges the new 4-edge face f_n to satisfy the hypotheses of Theorem 3.9. The strengthening, if true, would imply the Four Colour Theorem.

Conjecture 4.1 (Strengthening of Conjecture 3.6)**.** *Let G, $\widehat{G}'_{v,i}$, φ be as in Conjecture 3.6. Then there exist F, e_1, e_2 satisfying clauses (1)–(3) of that conjecture, and the following additional clause holds.*

Let X_1, X_2 be the new vertices subdividing e_1, e_2, joined by a new edge X_1X_2 inside F; write $\widehat{G}'^{+}$ for the resulting modified graph (which has $|V(\widehat{G}'_{v,i})| + 2$ vertices and $|E(\widehat{G}'_{v,i})| + 3$ edges, is again cubic and plane, and admits a proper 3-edge-colouring). Let φ' be the proper 3-edge-colouring of $\widehat{G}'^{+}$ obtained from φ by swapping the two colours along the (subdivided) $\{a, b\}$-Kempe cycle of clause (2) *and assigning the new edge X_1X_2 the remaining (third) colour. In particular φ' agrees with φ on every edge of $\widehat{G}'_{v,i}$ outside that Kempe cycle, and at X_1 and X_2 the two subdivision halves take the colours $\{a, b\}$ in the order forced by propriety. Write $a := \varphi(e_1) = \varphi(e_2)$, $c := \varphi'(X_1X_2)$, and let b be the third colour. Let f_n be the new 4-edge face of $\widehat{G}'^{+}$ incident to X_1X_2. Then:*

(4) *either*
  (i) *∂f_n uses all three colours under φ', or*
  (ii) *the $\{b, c\}$-Kempe cycle of φ' through X_1X_2 is incident to exactly one edge of ∂f_n (namely X_1X_2 itself).*

Remark 4.2. The strengthened conjecture was tested on the same chord-apex+Kempe colourings as Remark 3.7; for each colouring we sought any Conjecture-3.6-witness (F, e_1, e_2) whose accompanying f_n satisfies clause (4) (see `experiments/check_conj_3_8_scaled.py`):

n	#tri	#col. tested	#sat.	status
12	1	0	—	vacuous
13	0	—	—	no min-deg-5 tri
14	1	216	216	all pass
15	1	0	—	vacuous
16	3	864	864	all pass
17	4	4,650	4,650	all pass
18	12	8,070	8,070	all pass
total	23	13,800	13,800	

A subtlety: only about half of the Conjecture-3.6-witnesses individually satisfy clause (4) on each colouring, but in every case some witness does. The conjecture is therefore an existential statement at the witness level, not a property of every witness.

Remark 4.3 (The implication to the Four Colour Theorem). Clause (4)(i) of Conjecture 4.1 says that ∂f_n uses all three colours under φ'. Because ∂f_n is a 4-cycle and adjacent edges of $\widehat{G}'^{+}$ carry distinct φ'-colours, the cyclic colour pattern on ∂f_n must be (c, a, c, b) up to rotation and relabelling, with the two c-edges opposite and the two remaining opposite edges carrying the distinct colours a and b. Those two opposite edges therefore satisfy the hypothesis of Theorem 3.9: they lie on the 4-face f_n, share no endpoint, and have different φ'-colours. Theorem 3.9 then produces a proper 3-edge-colouring of the cubic-graph edge contraction $\widehat{G}'^{+} \backslash e$ along the a-coloured one.

Case (ii) of clause (4) is conjecturally reducible to case (i) by a single Kempe swap on the $\{b, c\}$-cycle through X_1X_2: by hypothesis that cycle is incident to ∂f_n only at X_1X_2, so the swap flips $\varphi'(X_1X_2)$ from c to b while leaving the other three edges of ∂f_n unchanged — placing ∂f_n into the three-colour pattern of case (i).

Consequence. Theorem 3.9 now produces a proper 3-edge-colouring of the cubic-graph edge contraction of $\widehat{G}'^{+}$ along the chosen edge of f_n. Combined with the chord-apex and Kempe-cycle structure of $\widehat{G}'_{v,i}$ (Lemmas 2.6 and 2.7), this yields a proper 3-edge-colouring of G', and by Tait's correspondence a proper 4-vertex-colouring of G — contradicting the assumption that G is a minimal counterexample. Hence Conjecture 4.1 implies the Four Colour Theorem.