# HEAWOOD RESTRICTIONS ON NESTED TIRE GRAPH DUALS

ERIC BAUERFELD

Abstract.

## 1. Introduction

A classical theorem of Tait recasts the Four Colour Theorem in dual, edge-colouring terms: a plane triangulation $G$ is properly 4-vertex-colourable if and only if its dual cubic graph $G'$ is properly 3-edge-colourable. Thus a minimal counterexample to the Four Colour Theorem – a smallest triangulation admitting no proper 4-colouring – corresponds to a smallest cubic plane graph admitting no proper 3-edge-colouring.

This paper continues the series studying that structure through the lens of *nested level duals*. The foundational vocabulary — level sources, levels, the inner planar dual $G'$ and its dual depth, and tire graphs — is developed in the companion paper [3]; we refer to that paper for those definitions and rely on them throughout. In particular we use, without restating, the notions of:

- *level source* $S$ and $G$-vertex levels $\ell_G(v)$;
- the inner planar dual $G'$ ([3, Definition 1.3]);
- *dual depth* $\delta_G(d_f)$ ([3, Definition 1.4]);
- *tire graph* $T = (B_{\text{out}}, O, E_{\text{ann}})$ with outer/inner boundaries and annular edges ([3, Definition 1.5]);
- the *tire-component lemma* ([3, Lemma 1.8]); and
- the *tire-tread partition theorem* ([3, Theorem 1.9]).

Throughout, $G = (V, E)$ is a plane maximal planar graph (a triangulation) with a fixed planar embedding $\Pi_G$. We write $|V| = n$, so $|E| = 3n - 6$ and $G$ has $2n - 4$ triangular faces.

The classical input is Heawood's face-sum identity [1]: for any proper 3-edge-colouring of a cubic plane graph $H$, assigning each face of $H$ a number in $\{+1, -1\}$ can be done so that the labels around every vertex of $H$ sum to 0 (mod 3). In the triangulation $G$ dual to $H$ this becomes a $\{+1, -1\}$ labelling of the *faces* of $G$ whose incident-face sum at every vertex of $G$ vanishes mod 3. Our aim is to record what this restriction forces along the boundary cycles of a nested tire graph, and to formulate a chain-pigeonhole programme in this Heawood labelling parallel to the medial programme of [4].

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## 2. Connected tire clusters

The tire treads at a fixed depth partition the depth-$d$ faces of $G$ [3], but distinct depth-$d$ tires need not be vertex-disjoint: a single vertex of $G$ may lie on the source-side boundary of several depth-$d$ tires at once (this occurs exactly when the depth-$d$ faces around that vertex are split into more than one arc by depth-$(d-1)$ faces). We organise the depth-$d$ tires by this sharing.

**Lemma 2.1** (Same-depth tires meet only in vertices)**.** *Let $T \neq T'$ be two distinct tire treads at the same depth $d$ in $\mathcal{T}(G,S)$, arising from connected components $C', C''$ of the depth-$d$ dual subgraph $G'_d$. Then $T$ and $T'$ share no edge of $G$; any intersection $V(T) \cap V(T')$ consists of isolated vertices.*

*Proof.* An edge $e$ of $G$ shared by two depth-$d$ annular faces $f_1, f_2$ is, by definition of the inner dual [3, Definition 1.3], a dual edge of $G'$ joining $d_{f_1}$ and $d_{f_2}$; since $\delta(d_{f_1}) = \delta(d_{f_2}) = d$, this edge lies in $G'_d$, so $d_{f_1}$ and $d_{f_2}$ belong to the same component of $G'_d$. Hence no edge of $G$ is shared by annular faces of two *different* components, and distinct depth-$d$ tires share no edge. Their intersection is therefore a set of isolated vertices. □

**Definition 2.2** (Connected tire cluster)**.** Fix a nested tire decomposition $\mathcal{T}(G,S)$ and a depth $d$. On the set of depth-$d$ tire treads define the relation

$$T \sim T' \iff V(T) \cap V(T') \neq \varnothing.$$

A *connected tire cluster* at depth $d$ is the subgraph of $G$

$$\mathsf{K} = \bigcup_i T_i \subseteq G$$

obtained as the union (of underlying plane graphs) of the tires in a single connected component $\{T_i\}$ of the transitive closure of $\sim$. A cluster consisting of a single tire is *trivial*; the connected tire clusters at depth $d$ partition the depth-$d$ tires.

*Remark* 2.3. By Lemma 2.1 the constituent tires of a connected tire cluster are joined only at shared vertices, each of which is a cut vertex of $\mathsf{K}$; a connected tire cluster is thus a “cactus of tires” and is in general *not* itself a tire graph, since the annulus structure of [3, Definition 1.5] fails at each such pinch. The shared (cut) vertices are precisely the vertices that belong to more than one depth-$d$ tire.

A single vertex may belong to several tires at one depth — the high-degree case where its depth-$d$ faces split into many arcs — so the number of *tires* through a vertex is unbounded. Clustering collapses exactly this multiplicity: all tires through a vertex at a fixed depth share that vertex, hence lie in one cluster. The cluster count is therefore controlled.

**Proposition 2.4** (A vertex meets at most two clusters)**.** *Every vertex $v \in V(G)$ belongs to at most two connected tire clusters, namely at most one at each of the two consecutive depths $\ell_G(v) - 1$ and $\ell_G(v)$. In particular a source vertex ($\ell_G(v) = 0$) belongs to a single cluster.*

*Proof.* Write $\ell = \ell_G(v)$.

*Step 1: every bounded face incident to $v$ has dual depth $\ell - 1$ or $\ell$.* Let $f$ be a bounded triangular face with $v \in V(f)$. Then $\delta_G(d_f) = \min_{u \in V(f)} \ell_G(u) \leq \ell_G(v) = \ell$. The other two vertices of $f$ are adjacent to $v$ in $G$, and the level function

$\ell_G(\cdot) = \mathrm{dist}_G(\cdot, S)$ is 1-Lipschitz along edges, so each has level at least $\ell - 1$; hence $\delta_G(d_f) \geq \ell - 1$. Thus $\delta_G(d_f) \in \{\ell-1, \ell\}$ (only $\delta_G(d_f) = 0$ when $\ell = 0$), so $v$ bounds faces of, and therefore belongs to tires of, no depth other than $\ell - 1$ or $\ell$.

*Step 2: at each depth, all tires through $v$ lie in one cluster.* Fix $d \in \{\ell - 1, \ell\}$ and let $T, T'$ be depth-$d$ tires with $v \in V(T) \cap V(T')$. Then $V(T) \cap V(T') \neq \varnothing$, so $T \sim T'$ in the sense of Definition 2.2, and all depth-$d$ tires containing $v$ lie in a single connected component of $\sim$ — one connected tire cluster $\mathsf{K}_d$.

Combining the two steps, $v$ belongs to at most the clusters $\mathsf{K}_{\ell-1}$ and $\mathsf{K}_\ell$, i.e. to at most two connected tire clusters; when $\ell = 0$ only $\mathsf{K}_0$ occurs. □

## 3. Heawood restrictions on the tire dual

We work inside a fixed nested tire decomposition $\mathcal{T}(G, S)$ of $G$ from a single-vertex level source $S$ [3], and use the tire data $T = (B_{\mathrm{out}}, O, E_{\mathrm{ann}})$ with annular faces $F_{\mathrm{ann}}$, outer boundary $B_{\mathrm{out}}$, and inner boundary $B_{\mathrm{in}}$ ([3, Definition 1.5]). Since $O$ is outerplanar, every vertex of a tire lies on $B_{\mathrm{out}}$ or on the inner-boundary walk $B_{\mathrm{in}}$; a tire has no interior vertices.

**Definition 3.1** (Heawood face-labelling of a tire)**.** A *Heawood face-labelling* of a tire graph $T$ is a map

$$\lambda : F_{\mathrm{ann}} \longrightarrow \{+1, -1\}$$

assigning a sign to each annular face of $T$. For a vertex $v \in V(T)$, write $F_{\mathrm{ann}}(v) \subseteq F_{\mathrm{ann}}$ for the set of annular faces of $T$ incident to $v$, and define the *induced vertex value*

$$\lambda^*(v) \;:=\; \sum_{f \in F_{\mathrm{ann}}(v)} \lambda(f) \mod 3 \;\in\; \{0, 1, -1\}.$$

The value $\lambda^*(v)$ is the *partial* face-sum at $v$ taken over the annular faces of $T$ alone, not over all faces of $G$ incident to $v$.

*Remark* 3.2. Because a tire has no interior vertices, every annular face of $T$ is incident to $B_{\mathrm{out}} \cup B_{\mathrm{in}}$, and a Heawood face-labelling is subject to *no* internal constraint: all $2^{|F_{\mathrm{ann}}|}$ sign assignments are admissible. The Heawood restriction is felt only on the two boundary cycles, through the induced vertex values $\lambda^*$.

**Definition 3.3** (Induced boundary sequences)**.** Let $\lambda$ be a Heawood face-labelling of $T$. Reading the vertices of $B_{\mathrm{out}}$ in clockwise order $v_0, v_1, \ldots, v_{p-1}$, the *outer Heawood sequence* of $(T, \lambda)$ is

$$\sigma_{\mathrm{out}}(T, \lambda) \;:=\; \big(\lambda^*(v_0), \ldots, \lambda^*(v_{p-1})\big) \;\in\; \{0, 1, -1\}^p.$$

Reading the inner-boundary walk $B_{\mathrm{in}}$ in clockwise order $w_0, \ldots, w_{q-1}$ gives the *inner Heawood sequence* $\sigma_{\mathrm{in}}(T, \lambda) \in \{0, 1, -1\}^q$. The *Heawood restriction relation* of $T$ is the set

$$R_T \;:=\; \big\{ \big(\sigma_{\mathrm{out}}(T, \lambda),\, \sigma_{\mathrm{in}}(T, \lambda)\big) \;:\; \lambda : F_{\mathrm{ann}} \to \{+1, -1\} \big\}$$

of all (outer, inner) sequence pairs realisable by a single face-labelling, read up to rotation and the global sign-flip $\lambda \mapsto -\lambda$ (equivalently $\sigma \mapsto -\sigma$).

**Definition 3.4** (Heawood compatibility across an interface)**.** Let $T$ be a tire and $T' \in \mathcal{T}(G, S)$ a child of $T$, so the outer boundary cycle $B_{\mathrm{out}}^{(T')}$ coincides with a bounded face of $O^{(T)}$; let $\gamma$ be this shared cycle, of length $L$, and let $v$ range over its vertices. Heawood face-labellings $\lambda$ of $T$ and $\lambda'$ of $T'$ are *compatible along* $\gamma$ if at every shared vertex $v$,

$$\lambda^*(v) + (\lambda')^*(v) \;\equiv\; 0 \pmod 3,$$

i.e. 0 is paired with 0 and +1 with −1. Equivalently, the inner Heawood sequence of $T$ on $\gamma$ is the pointwise negation mod 3 of the outer Heawood sequence of $T'$ on $\gamma$, after reversing one of the two clockwise readings to account for the opposite rotational senses in which $T$ and $T'$ traverse $\gamma$.

*Remark* 3.5. Call $v$ *interior* if it is not incident to the outer face of $\Pi_G$. For an interior vertex every incident face is bounded, and compatibility along $\gamma$ at $v$ is exactly the statement that the incident-face sum at $v$ — over the parent's annular faces together with the child's — vanishes mod 3:

$$\sum_{f \ni v} \lambda(f) \;\equiv\; 0 \pmod 3 \qquad \text{for every interior vertex } v \in V(G), \tag{3.1}$$

the sum ranging over the bounded faces incident to $v$. The interfaces of $\mathcal{T}(G, S)$ are interior level cycles, so cluster compatibility only ever constrains interior vertices and is untouched by the outer face.

To pass from (3.1) to a colouring one must account for the outer face: an outer-boundary vertex is incident to the unbounded face $f_\infty$, whose label is omitted from the bounded sum. Extend $\lambda$ by a single label $\lambda(f_\infty) \in \{+1, -1\}$ on $f_\infty$. Then a family of Heawood face-labellings that is pairwise compatible along every interface of $\mathcal{T}(G, S)$ assembles into a $\{+1, -1\}$ labelling of *all* faces of $G$ for which $\sum_{f \ni v} \lambda(f) \equiv 0 \pmod 3$ holds at every vertex — the outer-boundary vertices now carrying $\lambda(f_\infty)$ in their sum. This is Heawood's face-sum identity [1] for a proper 3-edge-colouring of the full cubic dual of $G$, hence (by Tait) a proper 4-vertex-colouring of $G$.

**Why the programme runs between nested clusters.** The vanishing condition (3.1) at a vertex $v$ is a constraint on the *full* face-star of $v$. To run a pigeonhole between two objects — a child and a parent — we need that full sum to split as exactly two one-sided contributions, so that each vertex label is the combination of a single child value and a single parent value. This is true at the level of connected tire clusters, and *false* at the level of individual tires. Extend a Heawood face-labelling to a connected tire cluster $\mathsf{K}$ by labelling every annular face of every tire of $\mathsf{K}$, and for $v \in V(\mathsf{K})$ write

$$\lambda^*_{\mathsf{K}}(v) \;:=\; \sum_{f} \lambda(f) \;\bmod 3,$$

the sum over the annular faces of $\mathsf{K}$ incident to $v$.

**Proposition 3.6** (Two-sided cluster decomposition at a vertex)**.** *Let $v \in V(G)$ have level $\ell = \ell_G(v)$, and let $\mathsf{K}_\ell$ and $\mathsf{K}_{\ell-1}$ be the at most two connected tire clusters containing $v$, of depths $\ell$ and $\ell - 1$ respectively (Proposition 2.4). Then the bounded faces of $G$ incident to $v$ partition into the annular faces of $\mathsf{K}_\ell$ at $v$ and the annular faces of $\mathsf{K}_{\ell-1}$ at $v$, and*

$$\sum_{f \ni v} \lambda(f) \;\equiv\; \lambda^*_{\mathsf{K}_\ell}(v) + \lambda^*_{\mathsf{K}_{\ell-1}}(v) \pmod 3.$$

*Each one-sided value* $\lambda^*_{\mathsf{K}_d}(v)$ *is the* complete *sum over all depth-d faces at v, so the Heawood condition* (3.1) *at v reads*

$$\lambda^*_{\mathsf{K}_\ell}(v) + \lambda^*_{\mathsf{K}_{\ell-1}}(v) \equiv 0 \pmod 3,$$

*a pairing between the single child cluster* $\mathsf{K}_\ell$ *and the single parent cluster* $\mathsf{K}_{\ell-1}$*. (When* $\ell = 0$*, or when v bounds no depth-*$\ell$ *face, only one term is present.)*

*Proof.* By Proposition 2.4 (Step 1) every bounded face incident to $v$ has depth $\ell-1$ or $\ell$, partitioning the incident faces by depth; by Step 2 all depth-$\ell$ faces at $v$ lie in the single cluster $\mathsf{K}_\ell$ and all depth-$(\ell-1)$ faces at $v$ in $\mathsf{K}_{\ell-1}$. Hence the depth-$\ell$ part is exactly the annular faces of $\mathsf{K}_\ell$ at $v$, the depth-$(\ell-1)$ part those of $\mathsf{K}_{\ell-1}$, and summing $\lambda$ over the two parts gives the identity; (3.1) is its vanishing. □

*Remark* 3.7 (Failure at the tire level). Proposition 3.6 is what makes the binary parent/child pairing possible, and it requires the cluster. A vertex $v$ may lie on many depth-$\ell$ tires — the unbounded case of Section 2 — and the per-tire value $\lambda^*(v)$ of Definition 3.1 then records only the faces of *one* tire at $v$, a fragment of $v$'s face-star. No single child tire carries the complete depth-$\ell$ sum, so the label $\sum_{f\ni v}\lambda(f)$ cannot be written as one child value plus one parent value, and per-tire compatibility (Definition 3.4) fails to assemble to (3.1). Clustering repairs this: Proposition 2.4 guarantees exactly one cluster meets $v$ on each side, so $\lambda^*_{\mathsf{K}_\ell}(v)$ is the complete child contribution and $\lambda^*_{\mathsf{K}_{\ell-1}}(v)$ the complete parent contribution. Every vertex label is then realised as the combination of a single child-cluster value with a single parent-cluster value, and the pigeonhole programme below chains *nested connected tire clusters* rather than individual tires.

We write $R_{\mathsf{K}}$ for the *cluster Heawood restriction relation*: the set of (outer, inner) boundary Heawood sequence pairs realisable by a face-labelling of $\mathsf{K}$, defined as in Definition 3.3 but with the outer and inner boundaries of the cluster and the complete one-sided values $\lambda^*_{\mathsf{K}}$ in place of a single tire's, read up to rotation and global sign-flip. By Proposition 3.6 two nested clusters are compatible along their shared interface exactly when the inner sequence of the parent is the pointwise negation mod 3 of the outer sequence of the child (after the orientation reversal of Definition 3.4).

**Conjecture 3.8** (Heawood chain-pigeonhole principle)**.** *There is a function* $N(k)$ *such that the following holds. Let*

$$\mathsf{K}_0 \supset \mathsf{K}_1 \supset \cdots \supset \mathsf{K}_{N(k)}$$

*be a nested chain of connected tire clusters in* $\mathcal{T}(G, S)$ *whose shared interfaces have length at most k. Then two adjacent cluster restriction relations* $R_{\mathsf{K}_i}, R_{\mathsf{K}_{i+1}}$ *in the chain admit compatible face-labellings along their shared interface, after rotation and global sign-flip. Equivalently, the chain contains a local gluing step that cannot be obstructed by disjoint Heawood boundary restrictions.*

**Conjecture 3.9** (Heawood cluster route to the Four Colour Theorem)**.** *For every plane triangulation G and every level source S, the cluster Heawood restriction relations* $\{R_{\mathsf{K}} : \mathsf{K}$ *a connected tire cluster*$\}$ *admit a selection of face-labellings that is compatible along every cluster interface. By Proposition 3.6 and Remark 3.5 this yields a* $\{+1, -1\}$ *face-labelling of G satisfying* (3.1)*, hence G is properly* 4*-vertex-colourable.*

## References

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