MEDIAL TIRE CUTS

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Abstract. Starting from the medial tire decomposition of a plane triangulation, we study the cuts that medial tires make in the full medial graph. We will show how to use medial tires to decompose the medial graph into a tree of three faces.

1. Introduction

This paper builds on the medial tire decomposition of [1]. For a plane triangulation G with fixed embedding we use freely the terminology and notation introduced there: the full medial graph $M(G)$, its decomposition into full medial tire graphs $\mathsf{M}(T)$ indexed by the treads T of the tire tree $\mathcal{T}(G, S)$ at a level source S, the annular medial cycle $A(T)$, and the boundary medial vertex sets.

We will show how to use medial tires to decompose the medial graph into a tree of three faces.

2. Cutting a full medial tire graph

We first describe a procedure that simultaneously *labels* and *cuts* a single full medial tire graph $\mathsf{M}(T)$ so that, after the cuts, the only faces are the outer face and 3-faces (triangles)—the teeth of [1]. The labelling assigns to each tooth an integer *walk depth*; the cuts break the cyclic adjacencies of the teeth so that what remains is a tree of 3-faces.

By a *cut* we mean the duplication of a single vertex of $\mathsf{M}(T)$: the vertex is split into two copies and the embedding is slit open along it (a planar unzip), separating the faces that meet only at that vertex. A cut therefore reduces the number of bounded faces that are not teeth.

Throughout we use the teeth, up and down teeth, apexes, bites, the annular medial cycle $A(T)$, and the auxiliary plane graph $B(T)$ of [1]. Each tooth is a 3-face of $\mathsf{M}(T)$, and the inner faces of $B(T)$ (the root face and the bite inner-gap faces) are the larger faces to be cut into teeth.

Definition 2.1 (Walk-depth labelling and cut)**.** Let $\mathsf{M}(T)$ be a full medial tire graph. Assign walk depths and cuts as follows.

(1) Pick an arbitrary up tooth, the *entry tooth*. It has walk depth d.
(2) Traverse all the teeth that bound the inner face incident to the entry tooth clockwise until we reach the entry tooth, incrementing the walk depth by 1 for each tooth traversed. (The *inner face incident to the entry tooth* is

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the inner face of $B(T)$ whose boundary contains the annular edge of $A(T)$ carrying the entry tooth.)

(3) When you reach the last tooth in the face, perform a *cut* by duplicating the annular vertex at which the traversal closes—the annular vertex of $A(T)$ shared by the last tooth and the entry tooth.

(4) Find the tooth t with the highest walk depth which is a member of a bite.

(5) If t is incident to a face F with unlabelled teeth, traverse the teeth in F starting from t in the direction of the tooth incident to t which is unlabelled, and increment the walk depth by 1 as you travel. (Here a tooth is *incident to t* when it shares an annular vertex of $A(T)$ with t.)

(6) Repeat steps (3)–(5) until all teeth have been labelled.

References

[1] E. Bauerfeld, *Medial Tire Decompositions of Plane Triangulations*, manuscript (math-research repository), 2026.