HEAWOOD RESTRICTIONS ON NESTED TIRE GRAPH DUALS

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Abstract.

1. Introduction

A classical theorem of Tait recasts the Four Colour Theorem in dual, edge-colouring terms: a plane triangulation G is properly 4-vertex-colourable if and only if its dual cubic graph G' is properly 3-edge-colourable. Thus a minimal counterexample to the Four Colour Theorem – a smallest triangulation admitting no proper 4-colouring – corresponds to a smallest cubic plane graph admitting no proper 3-edge-colouring.

This paper continues the series studying that structure through the lens of *nested level duals*. The foundational vocabulary — level sources, levels, the inner planar dual G' and its dual depth, and tire graphs — is developed in the companion paper [3]; we refer to that paper for those definitions and rely on them throughout. In particular we use, without restating, the notions of:

- *level source* S and G-vertex levels $\ell_G(v)$;
- the inner planar dual G' ([3, Definition 1.3]);
- *dual depth* $\delta_G(d_f)$ ([3, Definition 1.4]);
- *tire graph* $T = (B_{\text{out}}, O, E_{\text{ann}})$ with outer/inner boundaries and annular edges ([3, Definition 1.5]);
- the *tire-component lemma* ([3, Lemma 1.8]); and
- the *tire-tread partition theorem* ([3, Theorem 1.9]).

Throughout, $G = (V, E)$ is a plane maximal planar graph (a triangulation) with a fixed planar embedding Π_G. We write $|V| = n$, so $|E| = 3n - 6$ and G has $2n - 4$ triangular faces.

The classical input is Heawood's face-sum identity [1]: for any proper 3-edge-colouring of a cubic plane graph H, assigning each face of H a number in $\{+1, -1\}$ can be done so that the labels around every vertex of H sum to 0 (mod 3). In the triangulation G dual to H this becomes a $\{+1, -1\}$ labelling of the *faces* of G whose incident-face sum at every vertex of G vanishes mod 3. Our aim is to record what this restriction forces along the boundary cycles of a nested tire graph, and to formulate a chain-pigeonhole programme in this Heawood labelling parallel to the medial programme of [4].

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2. HEAWOOD RESTRICTIONS ON THE TIRE DUAL

We work inside a fixed nested tire decomposition $\mathcal{T}(G,S)$ of G from a single-vertex level source S [3], and use the tire data $T = (B_{\mathrm{out}}, O, E_{\mathrm{ann}})$ with annular faces F_{ann}, outer boundary B_{out}, and inner boundary B_{in} ([3, Definition 1.5]). Since O is outerplanar, every vertex of a tire lies on B_{out} or on the inner-boundary walk B_{in}; a tire has no interior vertices.

Definition 2.1 (Heawood face-labelling of a tire)**.** A *Heawood face-labelling* of a tire graph T is a map

$$\lambda : F_{\mathrm{ann}} \longrightarrow \{+1, -1\}$$

assigning a sign to each annular face of T. For a vertex $v \in V(T)$, write $F_{\mathrm{ann}}(v) \subseteq F_{\mathrm{ann}}$ for the set of annular faces of T incident to v, and define the *induced vertex value*

$$\lambda^*(v) \;:=\; \sum_{f \in F_{\mathrm{ann}}(v)} \lambda(f) \mod 3 \;\in\; \{0, 1, -1\}.$$

The value $\lambda^*(v)$ is the *partial* face-sum at v taken over the annular faces of T alone, not over all faces of G incident to v.

Remark 2.2. Because a tire has no interior vertices, every annular face of T is incident to $B_{\mathrm{out}} \cup B_{\mathrm{in}}$, and a Heawood face-labelling is subject to *no* internal constraint: all $2^{|F_{\mathrm{ann}}|}$ sign assignments are admissible. The Heawood restriction is felt only on the two boundary cycles, through the induced vertex values λ^*.

Definition 2.3 (Induced boundary sequences)**.** Let λ be a Heawood face-labelling of T. Reading the vertices of B_{out} in clockwise order $v_0, v_1, \ldots, v_{p-1}$, the *outer Heawood sequence* of (T, λ) is

$$\sigma_{\mathrm{out}}(T,\lambda) \;:=\; \big(\lambda^*(v_0), \ldots, \lambda^*(v_{p-1})\big) \;\in\; \{0,1,-1\}^p.$$

Reading the inner-boundary walk B_{in} in clockwise order $w_0, \ldots, w_{q-1}$ gives the *inner Heawood sequence* $\sigma_{\mathrm{in}}(T,\lambda) \in \{0,1,-1\}^q$. The *Heawood restriction relation* of T is the set

$$R_T \;:=\; \big\{ \big(\sigma_{\mathrm{out}}(T,\lambda),\, \sigma_{\mathrm{in}}(T,\lambda)\big) \;:\; \lambda : F_{\mathrm{ann}} \to \{+1,-1\} \big\}$$

of all (outer, inner) sequence pairs realisable by a single face-labelling, read up to rotation and the global sign-flip $\lambda \mapsto -\lambda$ (equivalently $\sigma \mapsto -\sigma$).

Definition 2.4 (Heawood compatibility across an interface)**.** Let T be a tire and $T' \in \mathcal{T}(G,S)$ a child of T, so the outer boundary cycle $B_{\mathrm{out}}^{(T')}$ coincides with a bounded face of $O^{(T)}$; let γ be this shared cycle, of length L, and let v range over its vertices. Heawood face-labellings λ of T and λ' of T' are *compatible along* γ if at every shared vertex v,

$$\lambda^*(v) + (\lambda')^*(v) \;\equiv\; 0 \pmod 3,$$

i.e. 0 is paired with 0 and +1 with −1. Equivalently, the inner Heawood sequence of T on γ is the pointwise negation mod 3 of the outer Heawood sequence of T' on γ, after reversing one of the two clockwise readings to account for the opposite rotational senses in which T and T' traverse γ.

Remark 2.5. Compatibility along γ at v is exactly the statement that the full incident-face sum at v — over the parent's annular faces together with the child's — vanishes mod 3:

$$\sum_{f \ni v} \lambda(f) \equiv 0 \pmod 3 \qquad \text{for every vertex } v \in V(G). \tag{2.1}$$

Since γ carries all faces of G incident to v between the two tires, a family of Heawood face-labellings that is pairwise compatible along every interface of $\mathcal{T}(G,S)$ assembles into a single $\{+1,-1\}$ face-labelling of G satisfying (2.1) at every vertex, hence (by Tait) a proper 4-vertex-colouring of G.

Conjecture 2.6 (Heawood chain-pigeonhole principle)**.** *There is a function $N(k)$ such that the following holds. Let*

$$T_0 \supset T_1 \supset \cdots \supset T_{N(k)}$$

be a nested chain of tires in $\mathcal{T}(G,S)$ whose shared interface cycles have length at most k. Then two adjacent Heawood restriction relations $R_{T_i}, R_{T_{i+1}}$ in the chain admit compatible face-labellings along their shared interface (Definition 2.4), after rotation and global sign-flip. Equivalently, the chain contains a local gluing step that cannot be obstructed by disjoint Heawood boundary restrictions.

Conjecture 2.7 (Heawood tire route to the Four Colour Theorem)**.** *For every plane triangulation G and every level source S, the Heawood restriction relations $\{R_T : T \in \mathcal{T}(G,S)\}$ admit a selection of face-labellings that is compatible along every interface of the tire tree. By Remark 2.5 this yields a $\{+1,-1\}$ face-labelling of G satisfying* (2.1)*, hence G is properly* 4*-vertex-colourable.*

References

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