MEDIAL TIRE DECOMPOSITIONS OF PLANE TRIANGULATIONS

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Abstract. We use the nested tire decomposition of a plane triangulation to induce a decomposition of its full medial graph into medial tire subgraphs. For a plane triangulation G, the medial graph $M(G)$ is naturally isomorphic to the medial graph of the planar dual G^*, and proper 3-vertex-colourings of $M(G)$ are equivalent to proper 3-edge-colourings of the cubic dual. Thus Tait's reformulation of the Four Colour Theorem may be studied through proper vertex 3-colourings of medial subgraphs. We define medial tire pieces, their boundary-state restriction relations, and a chain-pigeonhole conjecture for compatible medial boundary states across the tire tree.

1. Introduction

A classical theorem of Tait recasts the Four Colour Theorem in dual, edge-colouring terms: a plane triangulation G is properly 4-vertex-colourable if and only if its dual cubic graph G^* is properly 3-edge-colourable. The present paper records a medial version of this viewpoint. The vertices of the medial graph $M(G)$ correspond to edges of G, and adjacency in $M(G)$ records consecutiveness of edges around vertices and faces of G. Since planar duality interchanges vertices and faces while preserving the edge set, $M(G)$ is naturally isomorphic to $M(G^*)$.

Consequently a proper vertex 3-colouring of $M(G)$ is the same object as a proper edge 3-colouring of G^*. This suggests another route toward the Four Colour Theorem: rather than colouring the dual cubic graph directly, decompose the full medial graph into local annular pieces and try to prove that their proper vertex 3-colouring boundary restrictions always compose.

The structural input is the nested tire decomposition of [1]. A level source in a plane triangulation determines a rooted tree of tire treads. Each tread is an annular triangulated region with an outer boundary, an inner outerplanar graph, and annular triangular faces. We show that this decomposition induces a decomposition of $M(G)$ into medial tire subgraphs. The boundary data of a medial tire are proper 3-colourings of the medial vertices corresponding to boundary edges in the associated dual tire graph.

2. Background

Throughout, G is a simple plane maximal planar graph with fixed embedding, and G^* denotes its full planar dual. We use the level source, dual depth, tire graph, tire tread, and tire-tree terminology of [1]. In particular, a level source

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S determines a rooted tire tree $\mathcal{T}(G, S)$ whose vertices are tire treads and whose parent-child relation records nested containment across level-cycle interfaces.

Definition 2.1 (Medial graph)**.** Let H be a plane graph. The *medial graph* $M(H)$ has one vertex m_e for each edge $e \in E(H)$. Two medial vertices m_e, m_f are adjacent whenever e and f are consecutive in the cyclic order of edges around a vertex of H or around a face of H. The embedding is the standard one obtained by placing m_e at the midpoint of e and drawing medial edges through the vertex- and face-corners of H.

Remark 2.2. If H has bridges or vertices of degree 1, the usual medial construction may create parallel edges or loops depending on the chosen convention. In this paper the main application is to plane triangulations and their cubic planar duals, where the medial graph is a loopless 4-regular plane graph.

Proposition 2.3 (Medial dual invariance)**.** *Let H be a connected plane graph and let H^* be its planar dual. Then there is a natural plane-graph isomorphism*

$$M(H) \cong M(H^*).$$

Proof. Each edge $e \in E(H)$ corresponds to a unique dual edge $e^* \in E(H^*)$, giving a bijection $m_e \mapsto m_{e^*}$ between the vertices of $M(H)$ and $M(H^*)$. In $M(H)$ two vertices m_e, m_f are adjacent exactly when e and f are consecutive around either a vertex or a face of H. Under duality, vertices and faces are interchanged, and the cyclic order of the corresponding dual edges around the dual face or dual vertex is the same up to reversal. Thus the same pairs are medial-adjacent in $M(H^*)$, and the midpoint construction identifies the two embedded medial graphs. □

Corollary 2.4 (Tait colourings as medial vertex colourings)**.** *Let G be a simple plane triangulation. Proper vertex 3-colourings of $M(G)$ are in natural bijection with proper 3-edge-colourings of the cubic planar dual G^*.*

Proof. By Proposition 2.3, $M(G) \cong M(G^*)$. Vertices of $M(G^*)$ correspond to edges of G^*, and two such vertices are adjacent exactly when the corresponding dual edges are incident and consecutive around a vertex or face of G^*. Since G^* is cubic, proper vertex 3-colouring of $M(G^*)$ is therefore equivalent to assigning three colours to the edges of G^* so that the three edges incident to each dual vertex receive pairwise distinct colours. □

3. Medial tire pieces

Definition 3.1 (Full medial tire graph)**.** Let T be a tire tread in the tire tree $\mathcal{T}(G, S)$ supplied by [1]. The *full medial tire graph* of T, denoted $\mathsf{M}(T)$, is the subgraph of $M(G)$ induced by the medial vertices m_e with e an edge of G incident to at least one triangular face in the tread T. The medial vertices corresponding to annular edges of T are called *annular medial vertices.*

Remark 3.2. In the ambient-triangulation setting, the full medial tire graph $\mathsf{M}(T)$ coincides with the omitted-edge medial tire graph studied in [1]. Indeed, the medial edges of $\mathsf{M}(T)$ are contributed by corners of annular triangular tread faces. Such a face contains at most one outer-boundary edge and at most one inner-boundary edge, so it does not contribute a medial edge between two outer-boundary edges or between two inner-boundary edges. Similarly, chords of the inner outerplanar graph lie outside the annular tread and are not incident to annular tread faces.

Thus the deletion rule used for the earlier reduced medial tire graph removes no edges from the ambient object $\mathsf{M}(T)$.

The distinction only appears in the standalone drawing convention where the outer and inner boundary walks are added as artificial faces before forming a medial graph. Those artificial faces create same-boundary medial edges, and the reduced construction deletes them.

Theorem 3.3 (Annular medial colour bound)**.** *Let $T = (B_{\mathrm{out}}, O, E_{\mathrm{ann}})$ be a tire tread with non-degenerate boundaries and simple inner boundary B_{in}. Let $A(T)$ be the subgraph of $\mathsf{M}(T)$ induced by the annular medial vertices. For a graph H, write $\mathrm{Col}_3(H)$ for the set of proper 3-vertex-colourings of H. Then $A(T)$ is a cycle and*

$$|\,\mathrm{Col}_3(\mathsf{M}(T))| \;\leq\; |\,\mathrm{Col}_3(A(T))|.$$

Proof. Since the tread is a triangulated annulus with no vertices in its interior, each annular face has exactly one boundary edge, lying either on B_{out} or on B_{in}, and exactly two annular edges. As the annular faces are traversed cyclically around the tread, consecutive faces share one annular edge. Equivalently, the annular edges occur in a cyclic order in which each annular face contains two consecutive annular edges. Hence the subgraph of $\mathsf{M}(T)$ induced by the annular medial vertices is a cycle.

Consider the restriction map from proper 3-colourings of $\mathsf{M}(T)$ to colourings of this annular medial cycle $A(T)$. We claim that this map is injective. Let x be a non-annular medial vertex. Then x corresponds to an edge of B_{out} or B_{in}: chords of O are not incident to annular tread faces, and hence do not contribute vertices of $\mathsf{M}(T)$. This boundary edge is incident to a unique annular face of the tread, and the other two edges of that face are annular edges. Therefore x is adjacent in $\mathsf{M}(T)$ to the two annular medial vertices corresponding to those two annular edges.

Those two annular medial vertices are adjacent to each other, because their annular edges are consecutive on the same triangular annular face. In any proper 3-colouring they therefore receive two distinct colours, and x is forced to receive the remaining third colour. Thus every non-annular medial vertex has its colour uniquely determined by the colouring of $A(T)$. Two colourings of $\mathsf{M}(T)$ with the same restriction to $A(T)$ are identical, so the restriction map is injective. The stated inequality follows. □

Definition 3.4 (Boundary medial vertices)**.** Let T be a tire tread and let Γ_T be the corresponding dual tire subgraph in G^*. A vertex $m_e \in V(\mathsf{M}(T))$ is an *outer boundary medial vertex* if the corresponding dual edge $e^* \in E(G^*)$ lies on the outer boundary of Γ_T. It is an *inner boundary medial vertex* if e^* lies on the inner boundary of Γ_T. We write

$$\partial_{\mathrm{out}}\mathsf{M}(T) \quad \text{and} \quad \partial_{\mathrm{in}}\mathsf{M}(T)$$

for the two boundary sets.

Definition 3.5 (Medial tire restriction relation)**.** Let $\mathrm{Col}_3(X)$ denote the set of proper vertex 3-colourings of the induced subgraph on a vertex set X. The *medial tire restriction relation* of T is

$$R_T \subseteq \mathrm{Col}_3(\partial_{\mathrm{out}}\mathsf{M}(T)) \times \mathrm{Col}_3(\partial_{\mathrm{in}}\mathsf{M}(T)),$$

where $(\alpha, \beta) \in R_T$ exactly when $\alpha \cup \beta$ extends to a proper vertex 3-colouring of $\mathsf{M}(T)$.

Remark 3.6. The definition deliberately records boundary colourings on medial vertices corresponding to boundary edges in the dual tire graph. Under Corollary 2.4, these are precisely edge-colouring states on the boundary edges through which a dual tire piece meets its parent and children.

4. Decomposition

Corollary 4.1 (Medial tire decomposition)**.** *Let G be a plane triangulation with level source S. The tire-tree decomposition $\mathcal{T}(G,S)$ of* [1] *induces a rooted decomposition of the full medial graph $M(G)$ into full medial tire graphs $\{\mathsf{M}(T) : T \in V(\mathcal{T}(G,S))\}$, glued along their boundary medial vertex sets.*

Proof. By the tire-tread partition theorem of [1], the bounded triangular faces of G are partitioned into nested tire treads, with intersections between parent and child treads occurring only along their level-cycle interface data. Every edge of G that is incident to a bounded face therefore belongs to the closure of at least one tire tread, and an edge lying in two closures lies on the interface between adjacent treads in the tire tree. Passing to $M(G)$ sends edges of G to medial vertices. Thus each tread determines the induced subgraph $\mathsf{M}(T)$ on its incident edge set, and overlaps between two such subgraphs are exactly the medial vertices corresponding to interface edges, namely the appropriate boundary medial vertex sets. □

Definition 4.2 (Compatible family of medial tire colourings)**.** A *compatible family of medial tire colourings* on $\mathcal{T}(G,S)$ is a choice, for each tread T, of a proper vertex 3-colouring φ_T of $\mathsf{M}(T)$ such that whenever T' is a child tread of T, the two colourings agree on $V(\mathsf{M}(T)) \cap V(\mathsf{M}(T'))$.

Proposition 4.3 (Gluing criterion)**.** *The full medial graph $M(G)$ has a proper vertex 3-colouring if and only if the tire tree $\mathcal{T}(G,S)$ admits a compatible family of medial tire colourings.*

Proof. A proper vertex 3-colouring of $M(G)$ restricts to a proper vertex 3-colouring of every induced subgraph $\mathsf{M}(T)$, and these restrictions agree on overlaps.

Conversely, suppose a compatible family is given. Define a colour on each vertex m_e of $M(G)$ by choosing any tread T with $m_e \in V(\mathsf{M}(T))$ and setting $\varphi(m_e) = \varphi_T(m_e)$. Compatibility makes this independent of the choice of T. Every medial edge of $M(G)$ is drawn in a corner of some bounded triangular face of G or along the outer boundary interface. The relevant incident primal edges lie together in the closure of a single tire tread or in a shared boundary interface, where properness is already enforced by one of the local colourings. Hence φ is a proper vertex 3-colouring of $M(G)$. □

5. A medial pigeonhole programme

The restriction relation R_T records exactly the local information needed to pass a medial 3-colouring through a tire. In a nested chain

$$T_0 \supset T_1 \supset \cdots \supset T_k,$$

the outer boundary state of T_{i+1} must match an inner boundary state allowed by R_{T_i}. Thus a proof of the Four Colour Theorem in this framework would follow from a structural reason that these restriction sets cannot remain mutually disjoint along every branch of the tire tree.

Definition 5.1 (Medial boundary state)**.** A *medial boundary state* on a boundary set $\partial\mathsf{M}(T)$ is a proper vertex 3-colouring of the subgraph induced by that boundary set, considered up to permutation of the three colours and the dihedral symmetries of the boundary walk when that boundary is a cycle.

Conjecture 5.2 (Medial chain-pigeonhole principle)**.** *There is a function $N(k)$ such that the following holds. Let $T_0 \supset T_1 \supset \cdots \supset T_{N(k)}$ be a nested chain of tire treads whose relevant boundary medial walks have length at most k. Then two adjacent restriction relations in the chain have compatible medial boundary states after colour permutation and boundary symmetry. Equivalently, the chain contains a local gluing step that cannot be obstructed by disjoint proper vertex* 3*-colouring restrictions.*

Conjecture 5.3 (Medial tire route to the Four Colour Theorem)**.** *For every plane triangulation G and every level source S, the restriction relations $\{R_T : T \in V(\mathcal{T}(G,S))\}$ admit a compatible selection of boundary states across the tire tree. Hence $M(G)$ is properly vertex* 3*-colourable, G^* is properly* 3*-edge-colourable, and G is properly* 4*-vertex-colourable.*

Remark 5.4. Conjecture 5.3 is equivalent in strength to the Four Colour Theorem when combined with Tait's correspondence. The point of the formulation is not to weaken the target theorem, but to move the obstruction into finite boundary-state restrictions carried by annular medial tire pieces.

References

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