DUAL DECOMPOSITION OF MINIMAL COUNTEREXAMPLES

ERIC BAUERFELD

Abstract.

1. The minimal counterexample

Throughout, a *triangulation* is a simple plane graph, with a fixed embedding, in which every face — including the outer face — is bounded by a triangle. We first reduce to triangulations, then record the degree properties a smallest counterexample must have.

Lemma 1.1 (Reduction to triangulations)**.** *If every triangulation is properly* 4*-vertex-colourable, then so is every plane graph.*

Proof. Let H be a plane graph. Add edges to H, maintaining planarity, until no further edge can be added; the result is a triangulation H^+ on the same vertex set with $E(H) \subseteq E(H^+)$. A proper 4-colouring of H^+ restricts to a proper 4-colouring of H, since every edge of H is an edge of H^+. □

By Lemma 1.1, if the Four Colour Theorem fails then it fails for some triangulation. We may therefore make the following assumption.

Definition 1.2 (Minimal counterexample)**.** Let G be a triangulation on the fewest vertices that admits no proper 4-vertex-colouring. We call G a *minimal counterexample.* By minimality, every triangulation on fewer than $|V(G)|$ vertices is properly 4-colourable.

Remark 1.3. Since every triangulation on at most four vertices is properly 4-colourable (the largest being K_4), a minimal counterexample has $|V(G)| \geq 5$; the degree bound below sharpens this to $|V(G)| \geq 12$.

Lemma 1.4 (Minimum degree)**.** *A minimal counterexample* G *has minimum degree* $\delta(G) \geq 5$.

Proof. Suppose some vertex v has $\deg(v) = d \leq 4$.

If $d \leq 3$, let $G' = G - v$. Then G' is a plane graph on fewer vertices, so by Definition 1.2 and Lemma 1.1 it has a proper 4-colouring. The at most three neighbours of v use at most three colours, so a fourth colour is free for v, extending the colouring to G — a contradiction.

If $d = 4$, again 4-colour $G - v$. If the four neighbours of v use at most three colours we extend as before, so assume they receive all four colours; let v_1, v_2, v_3, v_4 be the neighbours in cyclic order around v, coloured $1, 2, 3, 4$. Consider the subgraph

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induced by the colour classes 1 and 3, and let K be its connected component containing v_1. If $v_3 \notin K$, swap colours 1 and 3 on K; now no neighbour of v is coloured 1, freeing it for v. If $v_3 \in K$, then a 1–3 Kempe chain joins v_1 to v_3, and this chain together with v encloses exactly one of v_2, v_4; hence the 2–4 component containing v_2 cannot also reach v_4, and swapping colours 2 and 4 on it frees colour 2 for v. Either way the colouring extends to G, a contradiction.

Hence $\delta(G) \geq 5$. □

2. The reduced dual

Write G' for the dual of G: since G is a triangulation, G' is a cubic plane graph in which each vertex of G corresponds to a face of G', each face of G to a vertex of G', and each edge to a dual edge. A vertex of G of degree k corresponds to a k-gonal face of G'.

By Lemma 1.4, $\delta(G) \geq 5$, and Euler's formula gives $\sum_{u \in V(G)} (6 - \deg u) = 12$, so G has a vertex of degree exactly 5 (indeed at least twelve). Fix such a vertex v. Its dual face F_v is a pentagon, bounded by the five dual vertices corresponding to the five faces of G incident to v.

Definition 2.1 (Reduced dual)**.** Let v be a degree-5 vertex of G with pentagonal dual face F_v, and fix an index $i \in \{0, 1, 2, 3, 4\}$. The *reduced dual* $\widehat{G}'_{v,i}$ is the plane graph obtained from G' as follows.

1. Delete the five dual vertices on the boundary of F_v, together with all edges incident to them. Each deleted vertex is cubic, with two edges on ∂F_v and one edge leaving F_v; deleting the five boundary vertices therefore removes the five external edges as well, dropping their five outer endpoints from degree 3 to degree 2. These five degree-2 vertices lie on the boundary of a single face F of the resulting graph.
2. List the five degree-2 vertices in clockwise order around F as $A = (A_0, A_1, A_2, A_3, A_4)$.
3. Add a new vertex v_n and join it to A_i, A_{i+1}, and A_{i+2} (indices mod 5) by three new edges.
4. Add a new edge between A_{i+3} and A_{i+4} (indices mod 5).

Remark 2.2. Steps (3) and (4) restore cubicity: A_i, A_{i+1}, A_{i+2} each gain one edge to v_n and A_{i+3}, A_{i+4} each gain the new edge, so all five return to degree 3, and v_n has degree 3. Since $A_i, \ldots, A_{i+2}$ and A_{i+3}, A_{i+4} are each consecutive along ∂F, the new vertex and edge can be drawn inside F without crossings, so $\widehat{G}'_{v,i}$ is again a cubic plane graph. The construction depends on the choice of i up to the rotational symmetry of A.

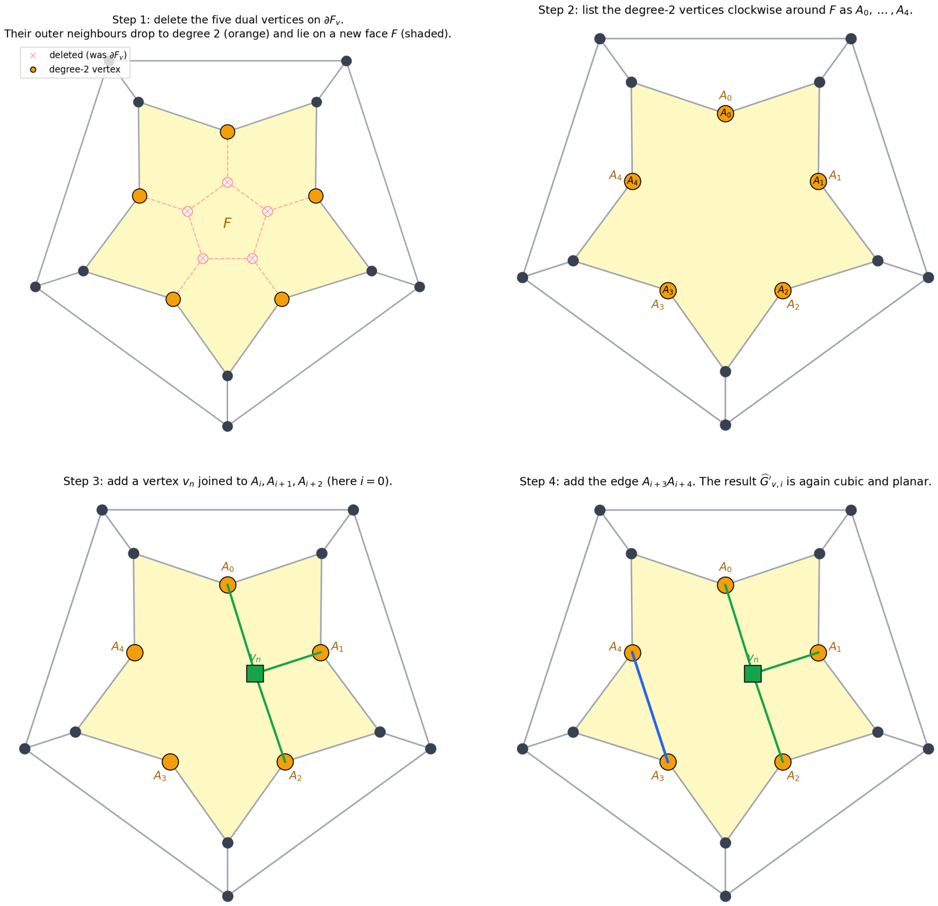

FIGURE 1. The four steps of Definition 2.1, illustrated on G' = the dodecahedron (dual of the icosahedron) with F_v the inner pentagon and $i = 0$. Top left: delete the five boundary vertices of F_v, leaving five degree-2 vertices on a new face F. Top right: order them clockwise as $A_0, \ldots, A_4$. Bottom left: add v_n joined to A_0, A_1, A_2. Bottom right: add the chord A_3A_4, giving the cubic plane graph $\widehat{G}'_{v,0}$.