MAXIMAL PLANAR GRAPH EDGE FLIPPING

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Abstract.

1. Motivation

The Four Color Theorem asserts that every planar graph is properly 4-colorable, or equivalently that no maximal planar graph G satisfies $\chi(G) \geq 5$. Suppose, towards a contradiction, that such a graph exists; let G_0 be one of minimum order. Any structural property shared by every maximal planar graph H with $|V(H)| = |V(G_0)|$ is then automatically inherited by G_0, and any property *not* satisfied by G_0 excludes a portion of the class of maximal planar graphs from playing the role of a minimum counterexample.

This paper investigates one such property: invariance under an admissible edge flip. We call a maximal planar graph G *flip-symmetric* when some admissible flip at an edge of G returns a graph isomorphic to G. Our principal observation (Theorem 4.1) is that a minimum-order 5-chromatic maximal planar graph cannot be flip-symmetric, so the search for a counterexample to the Four Color Theorem may, in principle, be confined to the complement of the class $\mathcal{F}$ of flip-symmetric graphs. This raises a quantitative question — how large is $\mathcal{F}$? — which we address empirically in Section 5 by an exhaustive census of maximal planar graphs of small order.

2. Preliminaries

Let G be a maximal planar graph with $|V(G)| \geq 4$, embedded in the plane so that every face — including the outer face — is a triangle. Every edge $uv \in E(G)$ is then shared by exactly two triangular faces uvw and uvx whose union is a quadrilateral $uwvx$ with diagonal uv.

Definition 2.1 (Edge flip)**.** Let G be a maximal planar graph and let $uv \in E(G)$ be an edge whose two incident triangular faces are uvw and uvx. The *edge flip* (or *diagonal flip*) at uv is the operation that deletes the edge uv and inserts the edge wx in its place, replacing the two triangles uvw and uvx by the two triangles uwx and vwx. The flip is *admissible* if $wx \notin E(G)$; otherwise the resulting multigraph is not simple and the flip is forbidden.

3. Flip-symmetric maximal planar graphs

For a maximal planar graph G and an admissible edge $uv \in E(G)$ with incident triangles uvw, uvx, write

$$G^{\mathrm{flip}(uv)} = \big(V(G),\, (E(G) \setminus \{uv\}) \cup \{wx\}\big)$$

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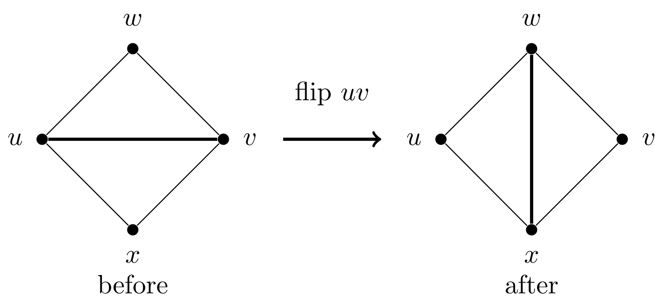

FIGURE 1. An edge flip replaces the diagonal uv of the quadrilateral $uwvx$ with the diagonal wx.

for the graph obtained from G by flipping uv.

Definition 3.1 (Flip-symmetric graph)**.** A maximal planar graph G is *flip-symmetric* if there exists an admissible edge $uv \in E(G)$ such that $G^{\mathrm{flip}(uv)} \cong G$. We write $\mathcal{F}$ for the class of flip-symmetric maximal planar graphs.

4. A minimal four-colorable counterexample

Theorem 4.1. *Let G be a maximal planar graph of minimum order among all maximal planar graphs H with $\chi(H) \geq 5$. Then $G \notin \mathcal{F}$; that is, G is not flip-symmetric.*

5. Flip symmetry frequency

To gauge how restrictive flip-symmetry is, we performed an exhaustive census of maximal planar graphs of small order. For each $n \in \{4, 5, \ldots, 14\}$ we enumerated every isomorphism class of maximal planar graph on n vertices using `plantri` (invoked through SageMath as `graphs.planar_graphs` with `minimum_connectivity` = 3 and `maximum_face_size` = 3), and for each such G we tested every admissible edge $uv \in E(G)$ for the existence of an isomorphism $G \cong G^{\mathrm{flip}(uv)}$. Writing T_n for the total number of maximal planar graphs on n vertices and $F_n = |\mathcal{F} \cap \{G : |V(G)| = n\}|$ for the number of flip-symmetric ones, the results are tabulated below.

n	T_n	F_n	F_n/T_n
4	1	0	0.000000
5	1	1	1.000000
6	2	1	0.500000
7	5	1	0.200000
8	14	5	0.357143
9	50	17	0.340000
10	233	48	0.206009
11	1,249	164	0.131305
12	7,595	552	0.072679
13	49,566	1,828	0.036880
14	339,722	6,164	0.018144

From $n = 10$ onward the ratio F_n/T_n decreases by a factor approaching $1/2$ at each step, suggesting that the density of flip-symmetric graphs among maximal planar graphs of order n decays to zero — empirically at a roughly geometric rate. In particular, the conclusion of Theorem 4.1 is consistent with the prevailing trend: as n grows, almost every maximal planar graph on n vertices is already excluded from flip-symmetry on purely structural grounds, and any putative counterexample to the Four Color Theorem is forced into a vanishingly small slice of the class.