LEVEL RESOLUTIONS OF MAXIMAL PLANAR GRAPHS

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Abstract. We propose a structural reformulation of the four color theorem in terms of *level resolutions* of maximal planar graphs. A level structure on a plane graph G is defined by BFS from a chosen level source (either a face or a degree-3 vertex), partitioning vertices into levels. A triangulation G' on the same vertex set is a *level resolution* of G from this source if the subgraphs of G' induced by even- and odd-level vertices are both bipartite. By construction, any level resolution admits an explicit 4-coloring obtained by 2-coloring each parity subgraph independently. The structural foundation of this approach is that each level subgraph L_k of G is outerplanar, and outerplanar graphs are 3-chromatic; the level-resolution problem is precisely to flip edges of G to reduce each L_k from chromatic number 3 to 2. We present computational results characterizing which isomorphism classes of maximal planar graphs on $n = 6, \ldots, 12$ vertices arise as level resolutions, and verify that every iso-class is reachable at every tested size.

1. Introduction

The four color theorem (4CT) asserts that every planar graph is 4-colorable. Equivalently, every maximal planar graph (triangulation) is 4-colorable. The Appel–Haken proof [1] and subsequent Robertson–Sanders–Seymour–Thomas refinement [2] rely on discharging arguments and computer-verified reducible configurations. Human-readable proofs remain elusive.

We propose a different structural approach. Given a plane triangulation G and a choice of *level source*, BFS from the source partitions the vertices into levels. A triangulation G' on the same vertex set is a *level resolution* of G if, when its vertices are labelled by the parity of their G-levels, the subgraph of G' induced by even-parity vertices and the subgraph induced by odd-parity vertices are both bipartite. The 4-coloring of G' then follows by definition: 2-color each parity subgraph and identify the four resulting classes with four distinct colors.

The remaining question is when level resolutions exist. We conjecture:

(i) every plane triangulation G' is a level resolution of some plane triangulation G via some level source; or, in a restricted form,
(ii) every plane triangulation of minimum degree at least 5 is a level resolution of some plane triangulation.

This paper formalizes the definitions and presents computational evidence bearing on (i)–(ii) for small vertex counts.

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2. Definitions

Throughout, $G = (V, E)$ is a plane maximal planar graph (a triangulation) with a fixed planar embedding Π_G. We write $|V| = n$, so $|E| = 3n - 6$ and G has $2n - 4$ triangular faces.

Definition 2.1 (Level source)**.** A *level source* of G is either:

- a face F of G (all vertices of F are level-0 sources), or
- a vertex v of degree 3 (the singleton $\{v\}$ is a level-0 source).

Definition 2.2 (Levels)**.** Given a level source $S \subseteq V$, the *level* of $v \in V$ is $\ell_G(v) = \mathrm{dist}_G(v, S)$, the graph distance from v to the nearest source vertex.

Definition 2.3 (Parity subgraph)**.** Let G be a triangulation with level source S, and let G' be a triangulation on the same vertex set as G. The *even parity subgraph* $E_{G,S}(G')$ is the subgraph of G' induced by $\{v \in V : \ell_G(v) \equiv 0 \pmod 2\}$. The *odd parity subgraph* is defined analogously for odd ℓ_G.

Definition 2.4 (Level resolution)**.** A triangulation G' on the same vertex set as G is a *level resolution* of G from level source S if both the even and odd parity subgraphs $E_{G,S}(G')$ and $O_{G,S}(G')$ are bipartite.

By construction, when G' is a level resolution of G via S, an explicit proper 4-coloring of G' is obtained by 2-coloring each parity subgraph independently (e.g., via BFS) and assigning the four resulting classes to distinct colors: even vertices receive red/blue, odd vertices receive yellow/green. The edges of G' partition into (i) edges within a parity subgraph, properly colored by the bipartition of that subgraph; and (ii) edges between an even-parity and odd-parity vertex, which connect disjoint color sets and so are properly colored.

3. Structural foundation: outerplanarity of level subgraphs

For each integer $k \geq 0$ and each (G, S), write L_k for the subgraph of G induced by the level-k vertices.

Theorem 3.1. *For every plane triangulation G and every level source S of G, each level subgraph L_k is outerplanar.*

Proof. For $k = 0$, L_0 is either a single vertex (when S is a degree-3 vertex) or the triangle bounding the source face (when S is a face), both outerplanar. Fix $k \geq 1$ and suppose, for contradiction, that L_k is not outerplanar.

Let D_k denote the planar drawing of L_k inherited from Π_G: that is, the set of points and curves in the plane representing the vertices and edges of L_k exactly as they appear in the embedding Π_G. Since L_k is not outerplanar, no face of D_k has every vertex of L_k on its boundary.

Let F^* be the face of D_k containing the source: when $S = \{v\}$, the face containing the point v; when S is a face F of G, the face containing the open region of F together with its three bounding vertices. The latter is well defined because each vertex of F lies at level 0 (hence is not a vertex of L_k) and each edge of F joins two level-0 vertices (hence is not an edge of L_k), so F and its boundary lie in a single component of $\mathbb{R}^2 \setminus D_k$. By assumption there exists $u \in L_k$ with $u \notin \partial F^*$.

Choose a BFS path $P : v_0, v_1, \ldots, v_k = u$ with $v_0 \in S$ and $v_i \in L_i$. For $0 \leq i \leq k - 1$, v_i lies in L_i and so is not a vertex of L_k; for $1 \leq i \leq k$, the edge

$v_{i-1}v_i$ joins L_{i-1} to L_i and so is not an edge of L_k. Hence, viewed as a curve in the plane, P meets the drawing D_k only at its endpoint u.

The complement $\mathbb{R}^2 \setminus D_k$ is open, and $P \setminus \{u\}$ is its continuous image of a connected set, hence lies in a single face of D_k. Since $v_0 \in F^*$, in fact $P \setminus \{u\} \subseteq F^*$, so $u \in \overline{F^*}$ and therefore $u \in \partial F^*$, contradicting the choice of u. □

The combinatorial significance of Theorem 3.1 is that outerplanar graphs are 3-chromatic [4]: their chromatic number is at most 3. Hence each L_k admits an independent 3-coloring, giving an immediate (but suboptimal) coloring of G using at most $3 \cdot \mathrm{depth}(G, S)$ colors when levels are colored independently. To recover a 4-coloring of G' via the parity-2-coloring strategy, what is required is to reduce each L_k's chromatic number from 3 to 2, equivalently to remove every odd cycle from each L_k:

Proposition 3.2. *If G' is a triangulation on the same vertex set as G such that for every k, the subgraph of G' induced by the level-k vertices of (G, S) is bipartite, and G' contains no edge between vertices at G-levels of equal parity and differing by exactly 2, then G' is a level resolution of G via S.*

Proof. The even parity subgraph $E_{G,S}(G')$ is the disjoint union of the even-level subgraphs of G' (since by hypothesis no edge of G' joins two even levels), each of which is bipartite. A disjoint union of bipartite graphs is bipartite. The same argument applies to the odd parity subgraph. □

This is the form of level resolution we seek to realize constructively: flips applied to G that break every odd cycle in every L_k without introducing cross-parity edges of distance 2.

4. The four-color conjecture via level resolutions

Conjecture 4.1 (Resolution preimage)**.** Every plane triangulation G' on n vertices is a level resolution of some plane triangulation G on n vertices.

If Conjecture 4.1 holds, the 4-coloring of any triangulation G' follows from the definition: exhibit a level-resolution preimage G, compute the BFS levels in G from the witness source, and 4-color G' via the parity 2-coloring.

5. Computational evidence

We enumerated all non-isomorphic triangulations on $n \in \{6, \ldots, 12\}$ via vertex insertion followed by edge-flip closure (see `triangulation_gen.py` and the faster `triangulation_gen_fast.py` for $n \geq 11$). For each isomorphism class, we computed the full set of iso-classes reachable as level resolutions across all valid level sources.

5.1. Coverage at $n = 6, \ldots, 12$.

Table 1 lists the resolution behavior for each iso-class. A class T_i is *covered* if it appears as the resolution iso-class of some triangulation.

Observation 5.1. For every $n \in \{6, \ldots, 12\}$, every plane-triangulation iso-class on n vertices is a level resolution of some plane triangulation on the same vertex set.

n	Iso-classes	Reachable as level resolutions
6	2	all 2
7	5	all 5
8	14	all 14
9	50	all 50
10	233	all 233
11	1249	all 1249
12	7595	all 7595

TABLE 1. Iso-class coverage under the level-resolution definition.

Equivalence to 4-colorability. A 2-partition $V = V_0 \sqcup V_1$ for which both $G'[V_0]$ and $G'[V_1]$ are bipartite induces a proper 4-coloring of G' (combine the bipartition of V_0 into colors $\{R, B\}$ and that of V_1 into $\{Y, G\}$), and conversely, any proper 4-coloring grouped pairwise produces such a partition. Hence by Definition 2.4, G' is a level resolution of some (G, S) if and only if G' admits a bipartite 2-partition of cardinality realizable as $(|V_e|, |V_o|)$ for some level source. Surjectivity at a given n is therefore equivalent to 4-colorability of every triangulation on n vertices together with realizability of the partition cardinality by some BFS. Our computational verification of Observation 5.1 does not invoke 4CT: we enumerate vertex partitions directly and check bipartiteness of the induced subgraphs.

5.2. **Surjectivity at $n = 12$: the icosahedron.** The icosahedron is the unique 5-regular triangulation on 12 vertices and a natural test case at $n = 12$: it is the smallest minimum-degree-5 plane triangulation, and its high symmetry constrains the achievable parity-cardinality splits to $(6, 6)$ from any source. We verify directly that the icosahedron admits a bipartite 2-partition of cardinality $(6, 6)$: with vertices labelled as in the standard icosahedral graph, the partition $\{0, 1, 2, 3, 4, 7\} \mid \{5, 6, 8, 9, 10, 11\}$ has both classes inducing bipartite subgraphs (each is a 6-cycle). By Definition 2.4, the icosahedron is therefore a level resolution of some plane triangulation on 12 vertices.

Observation 5.2. The icosahedron is a level resolution of some plane triangulation on 12 vertices.

5.3. **Restatement of the resolution-preimage conjecture.** In light of Observations 5.1 and 5.2, we restate Conjecture 4.1 in the form we will focus on through the constructive sections that follow:

Conjecture 5.3 (md_5 surjectivity)**.** For every $n \geq 12$, every minimum-degree-5 plane triangulation on n vertices is a level resolution of some plane triangulation on n vertices.

By the equivalence noted in Section 3, this is equivalent to a 4-coloring statement: every minimum-degree-5 plane triangulation admits a proper 4-coloring whose color-class cardinalities, grouped pairwise, match some BFS-level parity cardinality on the same vertex set. The md_5 restriction is the form most amenable to the constructive techniques explored in Section 6; the unrestricted preimage conjecture (Conjecture 4.1) appears to hold at every tested n as well, but md_5 removes the degree-3 and degree-4 vertices that obstruct several of the inductive structures considered later.

6. An edge-flip resolution algorithm

We describe an iterative edge-flip procedure aimed at producing, for a given (G, S), a triangulation G' on the same vertex set whose simple level cycles (with respect to the G-levels from S) are all even.

6.1. **Apex classification of L_k-edges.** Let $k \geq 1$. For each $uv \in E(L_k)$, the two triangles of G bounding uv have third vertices w, x, called the *apexes* of uv, with $\ell_G(w), \ell_G(x) \in \{k-1, k, k+1\}$ by BFS. We call uv *intra-level* when $\ell_G(w) = \ell_G(x) = k$, and *cross-level* otherwise.

Lemma 6.1. *If both apexes of $uv \in E(L_k)$ are at level $k-1$, then uv is a bridge of L_k.*

Proof sketch. In a plane triangulation, the neighbors of u in G at level $\leq k-1$ form a contiguous arc in the cyclic order around u. If both apexes w, x of uv lie at level $k-1$ on opposite sides of uv, then v lies in the complementary cyclic arc, which contains no other level-k neighbor of u. The symmetric statement around v gives that u is v's only level-k neighbor in the corresponding arc, so uv is a bridge of L_k. □

In particular every edge on a cycle of L_k has at least one apex at level k or $k+1$.

Proposition 6.2. *Flipping $uv \in E(L_k)$ with apexes w, x replaces uv with wx in G. The new edge wx belongs to L_k iff $\ell_G(w) = \ell_G(x) = k$, and to L_{k+1} iff $\ell_G(w) = \ell_G(x) = k+1$; otherwise wx is cross-parity and lies in no level subgraph. In all cases uv is removed from L_k.*

6.2. **Facial depth and isolated faces.**

Definition 6.3 (Facial depth)**.** Let L_k be drawn with the outerplanar embedding inherited from Π_G, let D be the dual graph of this drawing with the outer face removed, and let $\mathcal{B}$ be the set of inner faces incident to at least two edges of the outer face of L_k. The *facial depth* of an inner face F of L_k is

$$\operatorname{depth}(F) \;=\; \min_{F' \in \mathcal{B}} \operatorname{dist}_D(F, F'),$$

with the convention $\operatorname{depth}(F) = \infty$ if no such F' exists. An inner face is *isolated* if $\operatorname{depth}(F) \geq 1$.

The seed set $\mathcal{B}$ consists of the inner faces that already have two outer-face edges available as flip targets; Phase 2 below handles these directly. Phase 1 uses facial depth as a potential to push isolated odd faces toward $\mathcal{B}$.

A cycle that is *tricky-everywhere*, meaning every edge is intra-level, is necessarily isolated: an outer-face edge of L_k has a level-$(k-1)$ apex on its outer side and is therefore cross-level, so a tricky-everywhere cycle shares no edge with the outer face. Hence the tricky-everywhere cycles are a subset of the isolated odd cycles.

6.3. **Phase 1: interior faces. Procedure.** While some L_k contains an odd simple cycle whose corresponding inner face has facial depth ≥ 1 and shares no edge with the outer face, repeat:

(1) compute facial depths for all simple level cycles of L_k;
(2) among interior odd faces (depth ≥ 1, no outer-face edges) of maximum facial depth, pick one C; even-parity interior faces of depth ≥ 1 may also be selected as C;

(3) find the inner face F' incident to C of minimum facial depth, and flip the edge shared between C and F'.

The restriction to faces with no outer-face edge in step (2) means that every edge of C borders another inner face, so a unique shared-edge flip target exists for each neighbor F'. The depth-guided choice of F' in step (3) progressively pushes the residual odd-face structure toward the seed set $\mathcal{B}$ (depth 0). Even-face flips are optional restructuring moves that expand the reachable configuration space; the loop's termination is gated only by interior odd faces.

6.4. **Phase 2: outer-incident faces.** After Phase 1, every remaining odd simple cycle of L_k shares at least one edge with the outer face, whose apex pair includes a level-$(k-1)$ vertex and is therefore cross-level.

Procedure. For each L_k:

- every odd simple cycle $C \subseteq L_k$ incident to the outer face *must* have exactly one of its outer-face edges flipped;
- every even simple cycle of L_k incident to the outer face *may* have at most one of its outer-face edges flipped (an optional restructuring move).

For the source-face level ($k = 0$ with face source S), the L_0 source triangle is itself an odd cycle whose three edges all bound the outer face; we treat L_0 uniformly with higher levels, with the option of leaving the triangle intact when the resulting parity-subgraph configuration on G' permits.

Each flip is permitted even if the apex edge wx already exists in G, in which case G' is a multigraph; this does not affect bipartiteness of the parity subgraphs of G', since a duplicated edge is bipartite- equivalent to a single edge.

6.5. **Simple level resolutions.**

Definition 6.4 (Simple level resolution)**.** A plane triangulation G' is a *simple level resolution* of a plane triangulation G if there exists a level source S of G such that the algorithm of Sections 6–Phase 1 and Phase 2 applied to (G, S), under some sequence of optional-move choices, produces G' as a simple-graph triangulation whose parity subgraphs are bipartite.

6.6. **Empirical status.**

Observation 6.5. For every plane triangulation G on $n \in \{9, 10, 11\}$ vertices, every level source S, and every k such that L_k contains an odd simple cycle, the algorithm produces a G' whose corresponding L_k is bipartite (in the underlying simple-graph view). Across the 29640 such (G, S, k) triples — 4645 at $n \leq 10$ and 24995 at $n = 11$ — Phase 1 always terminates and Phase 2 always succeeds.

Coverage test. For each n we enumerate all plane-triangulation iso-classes on n vertices. For each iso-class G, each level source S of G, and each branching choice within the algorithm — Phase 1 ties on which deepest interior face and which lowest-depth neighbor to flip, Phase 2 choices of which outer-face edge to flip for each odd or even outer-incident cycle (including the option to leave even cycles untouched), and the option to skip the source-triangle break when S is a face — we run Phase 1 to termination and then Phase 2, recording the algorithm's output G' as a labelled simple graph. We check three properties: (a) G' is a triangulation (no multi-edge survived the Phase 2 flips), (b) the parity subgraphs $E_{G,S}(G')$ and $O_{G,S}(G')$ are both bipartite, and (c) the iso-class of G'. Aggregating over all

$(G, S, \text{branch-choices})$ triples yields the set of iso-classes attainable as algorithm outputs satisfying (a)+(b); we compare this set against the minimum-degree-5 iso-classes at each n.

Table 2 reports the empirical results of this test for $n = 6, \ldots, 11$.

n	Iso-classes	Algorithm output (any)	Simple level resolutions
6	2	all 2	all 2
7	5	all 5	4 of 5
8	14	all 14	12 of 14
9	50	all 50	49 of 50
10	233	231 of 233	225 of 233
11	1249	1247 of 1249	1232 of 1249

TABLE 2. Simple-resolution coverage under the algorithm of Section 6. "Algorithm output (any)" counts iso-classes that appear as the iso-class of some labelled triangulation output of the algorithm; "Simple level resolutions" counts iso-classes that additionally satisfy the bipartite-parity condition of Definition 6.4.

Observation 6.6. At $n = 12$ the unique minimum-degree-5 plane triangulation is the icosahedron. An exhaustive search over all 7595 iso-classes at $n = 12$, all level sources, and all algorithm branching choices found zero non-degenerate (G, S) pair such that the algorithm produces a triangulation isomorphic to the icosahedron with bipartite parity subgraphs. The icosahedron is nonetheless a level resolution in the sense of Definition 2.4 (Observation 5.2 and Table 1); the empirical evidence indicates that it is *not* a simple level resolution under the algorithm of Section 6.

The negative result of Observation 6.6 refutes the naive surjectivity statement that every md_5 triangulation is a simple level resolution; even the smallest example fails. Figure 1 illustrates the analogous obstruction at $n = 7$: the simplest missing iso-class. In both cases the source-face triangle of a face source forces an odd cycle in the even-parity subgraph, and a parallel small odd cycle appears in the odd-parity subgraph, so no choice of source S makes the target T its own level resolution; meanwhile no (G, S) with $G \neq T$ produces T under the algorithm either.

Question 6.7. Does Phase 1 terminate for all (G, S)? Equivalently, is there an explicit monovariant on L_k's face structure that strictly decreases on every Phase 1 flip?

7. Discussion and open questions

The computational results suggest the following:

(1) Conjecture 4.1 (resolution preimage) holds at every tested size: all iso-classes on $n \in \{6, \ldots, 12\}$ vertices arise as level resolutions (Observation 5.1; the $n = 12$ case includes the icosahedron, verified directly in Observation 5.2).

(2) Each level subgraph L_k of G is outerplanar (Theorem 3.1), so each L_k is 3-chromatic classically and independently of 4CT. The level-resolution problem reduces to flipping edges of G so that each L_k's chromatic number drops from 3 to 2, while avoiding creation of G-level-2 same-parity edges (Proposition 3.2).

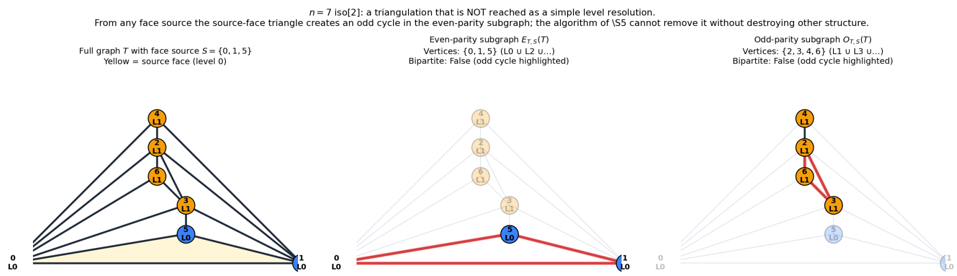

FIGURE 1. At $n = 7$, iso-class T with degree sequence $(6, 5, 5, 5, 3, 3, 3)$ is not a simple level resolution. Left: T drawn with face source $S = \{0, 1, 5\}$ outlined; vertices are labelled by their BFS level. Middle: the even-parity subgraph $E_{T,S}(T)$, induced on $\{0, 1, 5\}$, contains the source-face triangle (highlighted) — an odd cycle. Right: the odd-parity subgraph $O_{T,S}(T)$, induced on $\{2, 3, 4, 6\}$, contains a triangle on $\{2, 3, 6\}$ (highlighted). The obstruction is identical for every face source of T; vertex sources at degree-3 vertices produce L_1 triangles in the odd-parity subgraph.

(3) Under Definition 2.4, being a level resolution is equivalent to admitting a proper 4-coloring whose color cardinalities group pairwise to a BFS-realizable parity split. The structural framing through outerplanarity refines this: a constructive 4-coloring of G' is obtained via independent 2-colorings of each L_k in G', and the proof obligation is purely about removing odd cycles within outerplanar graphs by local edge flips, an operation that does not invoke 4CT.

The algorithm of Section 6 is the candidate constructive answer. Phase 1 iteratively flips the shared edge between the deepest interior odd face and its lowest-depth neighbor, pushing the residual odd-face structure toward the seed set $\mathcal{B}$ at depth 0, with optional even-face restructuring moves along the way; Phase 2 disposes of the remaining outer-incident odd cycles by flipping an outer-face edge each (and optionally an even outer-incident face), accepting a multigraph if the apex edge already exists. Observation 6.5 records that the algorithm terminates and succeeds at the level-bipartiteness layer on all 29640 tested (G, S, k) triples at $n \in \{9, 10, 11\}$. Observation 6.6 records that the icosahedron is empirically *not* a simple level resolution under the algorithm; the naive minimum-degree-5 surjectivity statement therefore fails at the first md_5 case. Table 2 shows that the gap appears already at $n = 7$ with one missing iso-class, growing to 17 missing at $n = 11$; Figure 1 illustrates the structural obstruction. Question 6.7 asks whether Phase 1 terminates in general.

8. Implementation

The code accompanying this paper consists of the following modules:

- `level_cycles.py`: core library for levels, level cycles, flip candidates, and resolution enumeration.
- `triangulation_gen.py`: enumeration of all non-isomorphic triangulations on n vertices via vertex-insertion plus flip closure.
- `coverage.py`: iso-class coverage reports with optional source and target filters.
- `balanced_layout.py`: a planar drawing routine that starts from a Tutte embedding and uses random-search optimization to equalize interior face areas while maintaining planarity.
- `four_color.py`: 4-coloring of G' via independent BFS 2-colorings of parity subgraphs.
- Visualization scripts: `plot_oct.py`, `n7_examples.py`, `four_color_viz.py`.

References

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