EVEN LEVEL GRAPH GENERATORS: A CONSTRUCTIVE CONJECTURE STRONGER THAN THE FOUR COLOR THEOREM

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Abstract. We investigate whether two constructive families of 4-colorable triangulations are, for each n, already rich enough to produce every maximal planar graph on n vertices. The first family is the *bridge-derived level graphs*: starting from an *Even Level Graph* – a triangulation all of whose level cycles, measured by breadth-first distance from a chosen source vertex, are even – we apply *bridge switches*, edge switches that never close a cycle in either parity subgraph. Every such graph is 4-colorable, inheriting the parity 2-coloring of an Even Level Graph. The second family is the *intertwining trees*: triangulations whose vertices split into two sets each inducing a tree, which are 4-colorable by coloring the two trees from disjoint pairs of colors. We conjecture that every maximal planar graph is a bridge-derived level graph, an intertwining tree, or both. Since both families are 4-colorable by construction, the conjecture implies the four color theorem for triangulations, and hence for all planar graphs; in fact it is *strictly stronger*, demanding not merely that a 4-coloring exist but that every triangulation be assembled by one of these two explicit constructions. A proof would therefore be a new, constructive proof of the four color theorem – and correspondingly the conjecture is at least as hard, and very likely harder, than that theorem. We show that a triangulation is an intertwining tree exactly when its dual is Hamiltonian, so every triangulation on at most 20 vertices is an intertwining tree and the first possible failures occur at $n = 21$, at the six duals of the Holton–McKay graphs. We verify that all six are bridge-derived level graphs, confirming the conjecture in its first nontrivial case. Pushing further, we identify by exhaustive generation the unique 44-vertex non-Hamiltonian *cyclically* 5*-connected* cubic planar graph – settling a uniqueness question Holton–McKay left open – whose 24-vertex 5-connected dual is the first test of the conjecture outside the 3-cut family; it too is a bridge-derived level graph, two bridge switches from an Even Level Graph.

1. Introduction

The four color theorem states that every planar graph is properly 4-colorable. It suffices to prove this for *maximal* planar graphs (triangulations), since every planar graph is a spanning subgraph of one. The known proofs proceed by reducing a hypothetical minimal counterexample, either through computer-checked unavoidable configurations or through discharging.

We take a constructive view. Instead of coloring an arbitrary triangulation, we ask which triangulations can be *assembled* by operations that manifestly preserve 4-colorability, and whether those operations reach all of them. We study two such

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constructions, and our motivating question is whether the two together are sufficient: does every maximal planar graph on n vertices arise from one of them?

The first construction builds on the *level* structure of a triangulation. Fixing a source vertex and taking breadth-first levels, an *Even Level Graph* (Definition 4.1) is a triangulation whose level cycles are all even; equivalently both of its parity subgraphs are bipartite, and a 2-coloring of each parity subgraph – two colors for the even-level vertices, two for the odd – is a proper 4-coloring (Theorem 4.2). From an Even Level Graph we generate further triangulations by edge switches; restricting to *bridge switches* (Definition 4.4), which add an edge to a parity subgraph only when it is a bridge there, guarantees that no new cycle – and in particular no odd cycle – ever appears in a parity subgraph. The resulting *bridge-derived level graphs* (Definition 4.5) therefore remain 4-colorable by the same parity coloring.

The second construction is purely combinatorial: an *intertwining tree* (Definition 4.6) is a triangulation whose vertex set partitions into two parts each inducing a tree. Coloring one tree from $\{1, 2\}$ and the other from $\{3, 4\}$ is a proper 4-coloring, since edges inside a part join differently-colored tree vertices and edges across the parts join the disjoint color sets.

Our central question is whether these two families exhaust all triangulations (Conjecture 4.8). As both families consist of 4-colorable graphs, an affirmative answer would constitute a constructive proof of the four color theorem for triangulations, and hence for all planar graphs.

We emphasize that the conjecture is a *stronger* statement than the four color theorem, not an equivalent reformulation of it. A proper 4-coloring with its colors grouped as $\{1, 2\} \mid \{3, 4\}$ is exactly a partition of the vertices into two parts each inducing a bipartite subgraph, so the four color theorem is precisely the assertion that every triangulation admits such a partition. The conjecture asserts strictly more: that the partition can be realized *constructively* – as the level parity of an Even Level Graph reached by bridge switches, or as a split into two induced *trees*. The four color theorem alone supplies neither construction; bridge-derivability in particular is a reachability condition well beyond the bare existence of a 4-coloring, so the conjecture implies the four color theorem but is not implied by it. A proof would accordingly be a new, constructive proof of the four color theorem, and the conjecture is at least as hard to settle – and, absent any structural characterization of the bridge-derived family, very likely harder.

We connect the two constructions through duality: a triangulation is an intertwining tree if and only if its dual is Hamiltonian (Theorem 4.7). Tait's conjecture – that every 3-connected cubic planar graph is Hamiltonian – fails first at 38 vertices, where Holton and McKay found exactly six counterexamples; dually, every triangulation on at most 20 vertices is an intertwining tree, and the first triangulations that are not are the six 21-vertex duals of the Holton–McKay graphs. These six are therefore the first nontrivial test of the conjecture, and we verify that all six are bridge-derived level graphs – each at most four bridge switches from an Even Level Graph, and two of them Even Level Graphs already.

2. Definitions

Throughout, $G = (V, E)$ is a plane maximal planar graph (a triangulation) with a fixed planar embedding Π_G. We write $|V| = n$, so $|E| = 3n - 6$ and G has $2n - 4$ triangular faces.

Definition 2.1 (Level source)**.** A *level source* of G is any vertex $v \in V$; we write $S = \{v\}$ for the level-0 source.

Definition 2.2 (Levels)**.** Given a level source $S \subseteq V$, the *level* of $v \in V$ is $\ell_G(v) = \mathrm{dist}_G(v, S)$, the graph distance from v to the nearest source vertex.

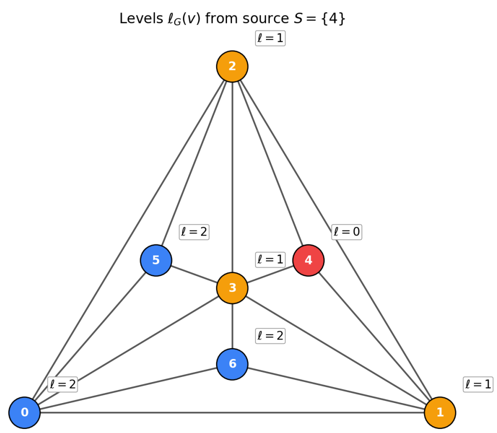

FIGURE 1. BFS levels from the degree-3 vertex source $S = \{4\}$. The source is level 0, its three neighbours are level 1, and the remaining vertices are level 2. Colour encodes the level.

Definition 2.3 (Level cycle)**.** A *level cycle* of G (with respect to a level source S) is a simple cycle in G all of whose vertices have the same level.

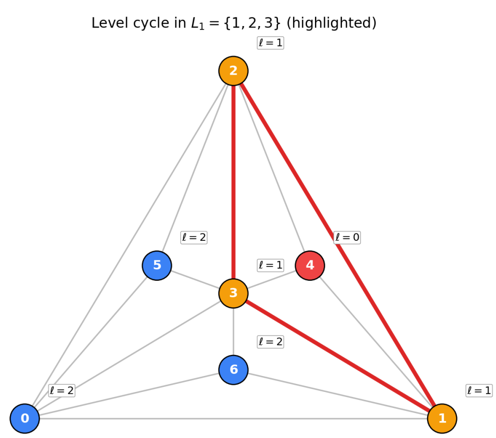

FIGURE 2. A level cycle in the triangulation of Figure 1. The triangle 1–2–3 is a simple cycle whose three vertices all lie at level 1, so it is a level cycle at level 1.

Definition 2.4 (Edge switch)**.** Let G be a triangulation with level source S, and let $e = uv$ be an edge of a level cycle of G. The *edge switch* at e is the edge flip on e: writing uvw and uvx for the two triangular faces of G containing e, the edge uv is removed and the edge wx is added. As with any edge flip, the result is a triangulation on the same vertex set provided w and x are non-adjacent in G.

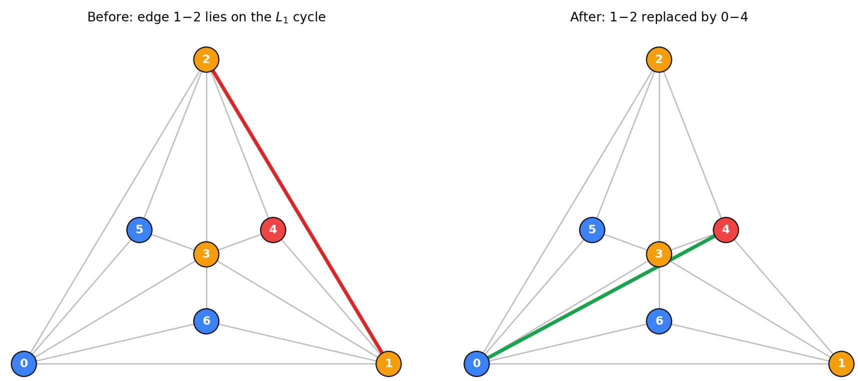

FIGURE 3. An edge switch on the level cycle of Figure 2. The chosen cycle edge $1-2$ is shared by the triangular faces $(0, 1, 2)$ and $(1, 2, 4)$; the switch deletes $1-2$ (red, left) and inserts $0-4$ (green, right). Vertex colours indicate the original levels in G.

Definition 2.5 (Parity subgraph)**.** Let G be a triangulation with level source S, and let G' be a triangulation on the same vertex set as G. The *even parity subgraph* $E_{G,S}(G')$ is the subgraph of G' induced by $\{v \in V : \ell_G(v) \equiv 0 \pmod 2\}$. The *odd parity subgraph* is defined analogously for odd ℓ_G.

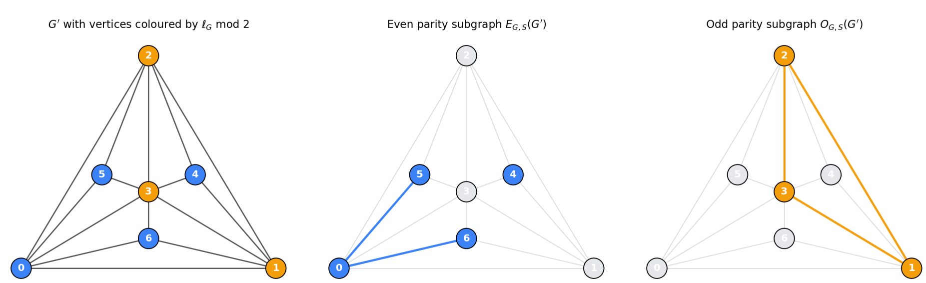

FIGURE 4. Parity subgraphs of $G' = T$ with respect to the level structure of Figure 1 (here we take $G = G' = T$). Left: T with vertices coloured by ℓ_G mod 2 (blue = even, orange = odd). Middle: the even parity subgraph $E_{G,S}(G')$, induced on $\{0, 4, 5, 6\}$; only edges with both endpoints even appear. Right: the odd parity subgraph $O_{G,S}(G')$, induced on $\{1, 2, 3\}$; the highlighted triangle shows that $O_{G,S}(G')$ is not bipartite for this choice of G'.

3. Outerplanarity of level components

For each integer $k \geq 0$ and each (G, S), write L_k for the subgraph of G induced by the level-k vertices. A *level component* of G (with respect to S) is a connected component of some L_k.

Theorem 3.1. *For every plane triangulation G and every level source S of G, every level component of G is outerplanar.*

Proof. Since every subgraph of an outerplanar graph is outerplanar, it suffices to show that each level subgraph L_k is outerplanar. For $k = 0$, $L_0 = S$ is a single vertex and is trivially outerplanar.

Fix $k \geq 1$ and let D_k be the drawing of L_k inherited from Π_G. Let F^* be the face of D_k containing the source. Suppose for contradiction that some $u \in L_k$ does not lie on ∂F^*, so u lies on the boundary of some other face of D_k. Take any path P in G from $v_0 \in S$ to u. As a curve in Π_G, P starts in F^* and ends at a point off ∂F^*, so it must transition from F^* to a different face of D_k; in a planar embedding this can happen only at a vertex of D_k, that is, at a level-k vertex w on P. Either $w \neq u$ (so P has length $\geq \mathrm{dist}_G(S, w) + 1 \geq k + 1$), or $w = u$ (contradicting $u \notin \partial F^*$). Since every S-to-u path has length $\geq k + 1$, $\mathrm{dist}_G(S, u) \geq k + 1$, contradicting $u \in L_k$. □

4. Even Level Graphs

Definition 4.1 (Even Level Graph)**.** A plane triangulation G with level source S is an *Even Level Graph* if every level cycle of G has even length.

Theorem 4.2. *Every Even Level Graph is* 4*-colorable.*

Proof. Since adjacent vertices in G have levels differing by at most 1, any edge between two same-parity endpoints in fact connects two vertices at the same level. Hence

$$E_{G,S}(G) \;=\; \bigsqcup_{i\geq 0} L_{2i}, \qquad O_{G,S}(G) \;=\; \bigsqcup_{i\geq 0} L_{2i+1},$$

and each L_k is bipartite because its cycles are level cycles of G, which have even length by hypothesis. Choose a 2-coloring of $E_{G,S}(G)$ in $\{\text{red}, \text{blue}\}$ and a 2-coloring of $O_{G,S}(G)$ in $\{\text{yellow}, \text{green}\}$. Same-parity edges of G are properly colored by the respective bipartition; opposite-parity edges connect $\{\text{red}, \text{blue}\}$ to $\{\text{yellow}, \text{green}\}$. The combined assignment is a proper 4-coloring of G. □

Enumeration for small n. Even Level Graphs are scarce among triangulations. For each n we enumerated all iso classes of plane triangulations and tested, for every choice of source vertex, whether all level subgraphs are bipartite (equivalently, whether every level cycle is even). Table 1 records, for $4 \leq n \leq 11$: the number of triangulation iso classes; how many of them admit at least one Even Level Graph source; the number of Even Level Graph iso classes (pairs (G, S) up to isomorphism, i.e. valid sources counted up to $\mathrm{Aut}(G)$); and the number of *flag-rooted* Even Level Graphs,

$$\sum_G \frac{4E}{|\mathrm{Aut}(G)|}\, s(G), \qquad E = 3n - 6,$$

where $s(G)$ is the number of valid sources of G. The flag-rooting is the automorphism-free count: $\mathrm{Aut}(G)$ acts freely on the $4E$ flags of a 3-connected triangulation, so every summand is an integer.

The smallest Even Level Graph is the octahedron at $n = 6$: from any vertex the four neighbours form a 4-cycle at level 1 and the antipode sits alone at level 2. Below $n = 6$ every triangulation forces an odd level cycle, so no Even Level Graph exists.

n	triangulations	with ELG source	ELG iso classes	flag-rooted ELGs
4	1	0	0	0
5	1	0	0	0
6	2	1	1	6
7	5	2	2	45
8	14	5	6	186
9	50	13	14	651
10	233	37	45	2766
11	1249	129	169	14346

TABLE 1. Even Level Graph counts for $4 \le n \le 11$. The *triangulations* column is the number of plane-triangulation iso classes (OEIS A000109). *ELG iso classes* counts pairs (G, S) up to isomorphism; *flag-rooted ELGs* is the automorphism-free count $\sum_G \frac{4E}{|\mathrm{Aut}(G)|}\, s(G)$.

Definition 4.3 (Derived level graph)**.** Let G be an Even Level Graph with level source S, and let E and O denote the edge sets of the even and odd parity subgraphs $E_{G,S}(G)$ and $O_{G,S}(G)$. A *derived level graph* of G is a triangulation G' on the same vertex set as G obtained by a sequence of edge switches (Definition 2.4), each acting on an edge of E or of O. We do not update E or O to reflect the level structure of intermediate triangulations: throughout the sequence, an edge is classified as belonging to E (resp. O) if and only if both of its endpoints have even (resp. odd) level in G.

A derived level graph G' is *valid* if both $E_{G,S}(G')$ and $O_{G,S}(G')$ contain only even cycles.

Definition 4.4 (Bridge switch)**.** Let G' be a triangulation reached from an Even Level Graph G, with parity classes inherited from G as in Definition 4.3. An edge switch on an edge $e \in E \cup O$ of G', replacing uvw, uvx by the edge wx, is a *bridge switch* if either

- the new edge wx is a cross-parity edge (one endpoint even, the other odd), so wx enters neither parity subgraph; or
- wx is a same-parity edge and is a *bridge* in the parity subgraph it joins – that is, w and x lie in different connected components of that parity subgraph, so adding wx creates no new cycle.

Definition 4.5 (Bridge-derived level graph)**.** A *bridge-derived level graph* of an Even Level Graph G is a triangulation obtained from G by a sequence of bridge switches (Definition 4.4).

Because a bridge switch never closes a cycle in a parity subgraph, it never introduces an odd cycle there. As an Even Level Graph has bipartite parity subgraphs (every level cycle is even), every bridge-derived level graph has bipartite parity subgraphs as well, and so is automatically a valid derived level graph. Equivalently, the first Betti number of each parity subgraph is non-increasing along any sequence of bridge switches.

Definition 4.6 (Intertwining tree)**.** A maximal planar graph G is an *intertwining tree* if its vertex set can be partitioned into two sets A and B such that both induced subgraphs $G[A]$ and $G[B]$ are trees.

Theorem 4.7. *A maximal planar graph G is an intertwining tree if and only if its dual G^* has a Hamiltonian cycle.*

Proof. ($\Rightarrow$) Let $V(G) = A \sqcup B$ with $G[A]$ and $G[B]$ trees. Every triangular face $\{x, y, z\}$ of G meets both A and B: if all three vertices were in A the triangle would be a cycle in the tree $G[A]$, and likewise for B. Draw a closed curve through the faces of G separating the A-vertices from the B-vertices within each face. Since every face is split, the curve visits every face exactly once and crosses an edge of G precisely when that edge joins A to B; it is therefore a Hamiltonian cycle of G^*.

($\Leftarrow$) Let H be a Hamiltonian cycle of G^*. Drawn in the plane, H is a Jordan curve visiting every face of G once; let A and B be the vertices of G interior and exterior to H. The $2n - 4$ edges of H cross exactly the edges of G between A and B, leaving $(3n - 6) - (2n - 4) = n - 2$ edges inside $G[A]$ and $G[B]$ together. The edges inside A lie in the disk bounded by H and span A without enclosing a face (each face is cut by H), so $G[A]$ is a tree; likewise $G[B]$. □

Conjecture 4.8. Every maximal planar graph is a bridge-derived level graph of some Even Level Graph, an intertwining tree, or both.

Since a bridge-derived level graph is automatically a valid derived level graph, this is a stronger statement than the corresponding conjecture phrased with arbitrary E/O switches; it is also the form that the evidence below actually supports.

By Theorem 4.7, the intertwining-tree disjunct fails for G exactly when G^* is a counterexample to Tait's conjecture. The smallest such G^* have 38 vertices (Holton–McKay [1], exactly 6 graphs), so the smallest triangulations that are not intertwining trees occur at $n = 21$ and there are exactly 6 of them. Below $n = 21$ every maximal planar graph is an intertwining tree, which is why the disjunction holds trivially in that range.

The boundary case $n = 21$. The first triangulations that are *not* intertwining trees are the six duals of the Holton–McKay graphs, at $n = 21$. For the disjunction to survive at $n = 21$, each of these six must be a valid derived level graph. We find:

- All six duals are confirmed not intertwining trees (exhaustive check of all $2^{20} - 1$ vertex bipartitions), consistent with Theorem 4.7.
- Two of the six are themselves Even Level Graphs (for a suitable source vertex), hence trivially valid derived level graphs. So the disjunction holds for them through the derived-level-graph disjunct – the first instances where that disjunct does work the intertwining-tree disjunct cannot.
- The remaining four are not Even Level Graphs for any source, and their full E/O-orbits ($\sim 10^8$ states per source labelling) are far too large to exhaust.

Restricting to *bridge switches* (Definition 4.4) shrinks the relevant orbits by roughly two orders of magnitude and, crucially, keeps every reachable triangulation valid. A backward bridge-switch search over the valid parity partitions found an Even Level Graph witness for each of the four, so all four are *bridge-derived level graphs* (Definition 4.5) and hence valid derived level graphs. The witnessing orbits are small – between a few hundred and $\sim 1.7 \times 10^5$ states – even though other parity partitions of the same triangulations have orbits exceeding 10^6; finding one good partition suffices. Each witness is in fact only a *handful* of bridge switches from its dual: the explicit Even Level Graph, parity labelling, and bridge-switch sequence are recorded for all six – path lengths $3, 1, 2, 4$ for these four and 0 for the two that are Even Level Graphs outright – and each step has been verified to be a valid bridge switch.

Thus at $n = 21$ the disjunction is confirmed for all six critical iso classes: two are Even Level Graphs outright, and the other four are bridge-derived level graphs. The bridge-switch restriction is what made the search tractable – it both shrinks the orbit and guarantees validity, so any Even Level Graph located in a backward orbit is an immediate witness. Table 2 records the outcome for each dual.

dual	intertwining tree	Even Level Graph source	bridge switches to ELG
0	no	–	3
1	no	10	0
2	no	9	0
3	no	–	1
4	no	–	2
5	no	–	4

Table 2. The six Holton–McKay duals at $n = 21$, the first triangulations that are not intertwining trees. Each is a bridge-derived level graph: duals 1 and 2 are Even Level Graphs outright (zero switches), and the remaining four reach an Even Level Graph in 1–4 bridge switches. All witnesses are step-verified.

The cyclically-5-connected case: $n = 24$. The six $n = 21$ duals all carry nontrivial 3-cuts in the cubic picture; dually, each contains a separating triangle, so each is built from smaller pieces and lies in the most reducible part of the non-Hamiltonian world. (The famous 46-vertex Tutte graph is no improvement here: it too is only cyclically 3-connected, and its 25-vertex dual has separating triangles.) The genuinely new regime is the *cyclically* 5*-connected* one, dual to a 5-connected triangulation – no separating 3- or 4-cycle, hence nothing to decompose along. By Holton–McKay, the smallest non-Hamiltonian cyclically 5-connected cubic planar graph has 44 vertices (Fig. 2.10 of [1], attributed to Tutte; minimality due to Faulkner–Younger), and its dual is a 24-vertex 5-connected triangulation.

We obtain this graph by generation rather than transcription. A 44-vertex cubic planar graph is the dual of a 24-vertex triangulation, and a cubic graph is cyclically 5-connected if and only if its dual triangulation is 5-connected. Enumerating all 5-connected triangulations on 24 vertices (`plantri -c5`, 6833 of them) and testing each dual for Hamiltonicity, we find that *exactly one* has a non-Hamiltonian dual.

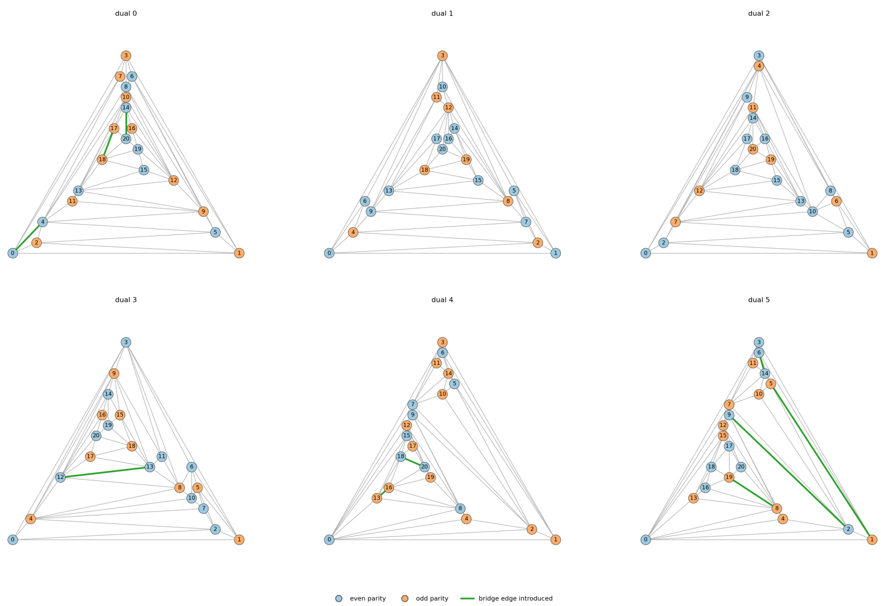

FIGURE 5. The six Holton–McKay duals, drawn as crossing-free planar graphs and coloured by parity (blue even, orange odd, with respect to the fixed level-parity labelling). The solid green edges are the bridge edges introduced by the bridge switches from each dual's witness Even Level Graph. Each green edge is a bridge of its parity subgraph, so no new cycle – and in particular no odd cycle – is created; duals 1 and 2 coincide with their Even Level Graphs and have no added edge.

This both produces the graph and, granting the correctness of the generator and the Hamiltonicity test, settles the uniqueness question Holton–McKay left open: there is a unique non-Hamiltonian cyclically 5-connected cubic planar graph on 44 vertices.

Let T be its dual: a 24-vertex triangulation with vertex connectivity 5 and no separating triangle, and – since its dual is non-Hamiltonian – not an intertwining tree. We find that T is nonetheless a bridge-derived level graph. Of its 333 valid parity partitions most are useless: their backward bridge-orbits exceed 8×10^5 states with no Even Level Graph in sight. But one partition has a backward orbit of only 4678 states containing an Even Level Graph (source $s = 19$, maximum level 4) at depth 2. The two bridge switches carrying that Even Level Graph to T are

$$\text{remove } \{16, 21\},\ \text{add } \{20, 22\} \quad \text{and} \quad \text{remove } \{15, 18\},\ \text{add } \{6, 19\},$$

each adding a same-parity edge that is a bridge of the (odd, resp. even) parity subgraph; both steps have been verified to be valid bridge switches. So the disjunction holds for T through the bridge-derived disjunct, and the "one good partition suffices" phenomenon seen at $n = 21$ persists into the cyclically 5-connected regime – the first test of the conjecture genuinely outside the 3-cut family.

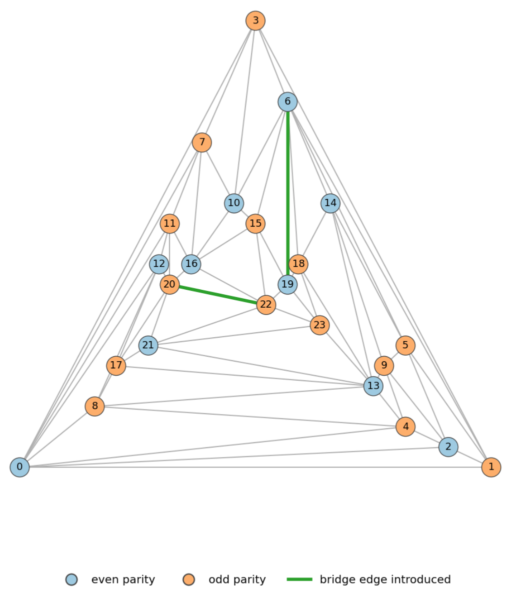

FIGURE 6. The 24-vertex dual T of the unique 44-vertex non-Hamiltonian cyclically 5-connected cubic planar graph (Holton–McKay Fig. 2.10), drawn crossing-free and coloured by the fixed parity labelling (blue even, orange odd). T is 5-connected and not an intertwining tree, yet is a bridge-derived level graph: the two solid green edges $\{6, 19\}$ and $\{20, 22\}$ are the bridge edges introduced by the two bridge switches carrying its witness Even Level Graph (source 19) to T. Each green edge is a bridge of its parity subgraph – $\{6, 19\}$ in the even subgraph, $\{20, 22\}$ in the odd – so no new cycle, and in particular no odd cycle, is created.

REFERENCES

[1] D. A. Holton and B. D. McKay. *The smallest non-Hamiltonian 3-connected cubic planar graphs have 38 vertices.* Journal of Combinatorial Theory, Series B, 45(3):305–319, 1988.