# LEVEL SWITCHING

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ABSTRACT.

## 1. INTRODUCTION

## 2. DEFINITIONS

Throughout, $G = (V, E)$ is a plane maximal planar graph (a triangulation) with a fixed planar embedding $\Pi_G$. We write $|V| = n$, so $|E| = 3n - 6$ and $G$ has $2n - 4$ triangular faces.

**Definition 2.1** (Level source)**.** A *level source* of $G$ is either:

- a face $F$ of $G$ (all vertices of $F$ are level-0 sources), or
- a vertex $v$ of degree 3 (the singleton $\{v\}$ is a level-0 source).

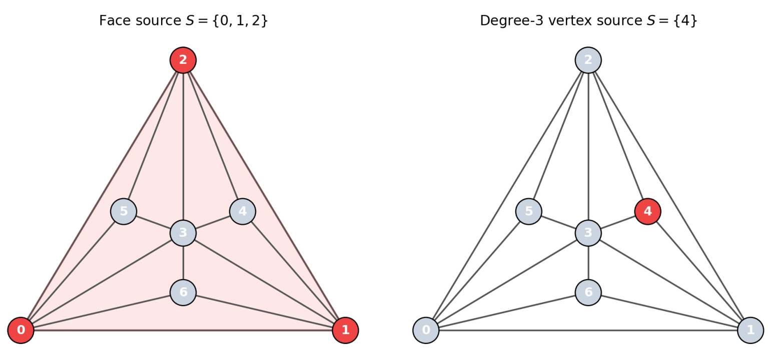


FIGURE 1. The two kinds of level source on a 7-vertex triangulation $T$ ($K_4$ with vertices $4, 5, 6$ stacked into the three interior faces). Left: the face source $S = \{0, 1, 2\}$ (level-0 vertices are the corners of the highlighted triangle). Right: the degree-3 vertex source $S = \{4\}$.

**Definition 2.2** (Levels)**.** Given a level source $S \subseteq V$, the *level* of $v \in V$ is $\ell_G(v) = \mathrm{dist}_G(v, S)$, the graph distance from $v$ to the nearest source vertex.

**Definition 2.3** (Level cycle)**.** A *level cycle* of $G$ (with respect to a level source $S$) is a simple cycle in $G$ all of whose vertices have the same level.

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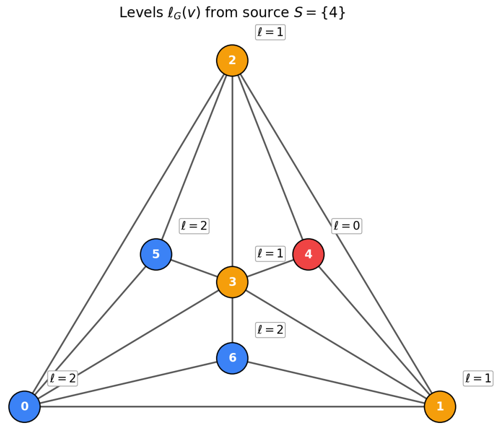


FIGURE 2. BFS levels from the degree-3 vertex source $S = \{4\}$. The source is level 0, its three neighbours are level 1, and the remaining vertices are level 2. Colour encodes the level.

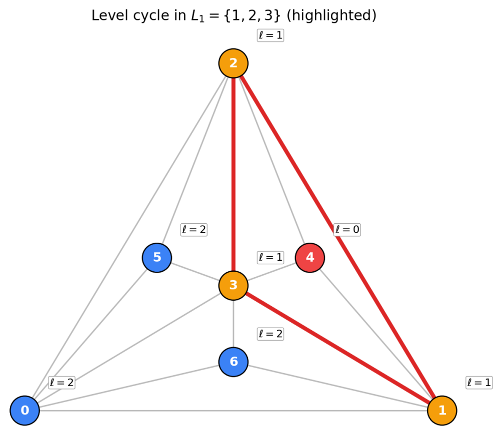


FIGURE 3. A level cycle in the triangulation of Figure 2. The triangle 1–2–3 is a simple cycle whose three vertices all lie at level 1, so it is a level cycle at level 1.

**Definition 2.4** (Edge switch)**.** Let $G$ be a triangulation with level source $S$, and let $e = uv$ be an edge of a level cycle of $G$. The *edge switch* at $e$ is the edge flip on $e$: writing $uvw$ and $uvx$ for the two triangular faces of $G$ containing $e$, the edge $uv$ is removed and the edge $wx$ is added. As with any edge flip, the result is a triangulation on the same vertex set provided $w$ and $x$ are non-adjacent in $G$.

**Definition 2.5** (Parity subgraph)**.** Let $G$ be a triangulation with level source $S$, and let $G'$ be a triangulation on the same vertex set as $G$. The *even parity subgraph* $E_{G,S}(G')$ is the subgraph of $G'$ induced by $\{v \in V : \ell_G(v) \equiv 0 \pmod 2\}$. The *odd parity subgraph* is defined analogously for odd $\ell_G$.

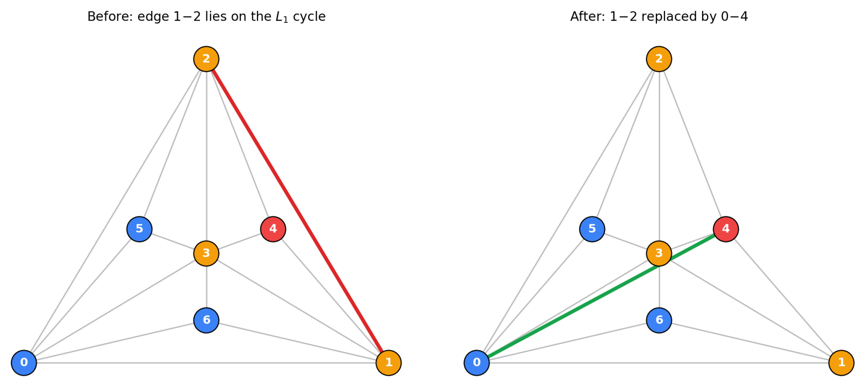


FIGURE 4. An edge switch on the level cycle of Figure 3. The chosen cycle edge $1-2$ is shared by the triangular faces $(0,1,2)$ and $(1,2,4)$; the switch deletes $1-2$ (red, left) and inserts $0-4$ (green, right). Vertex colours indicate the original levels in $G$.

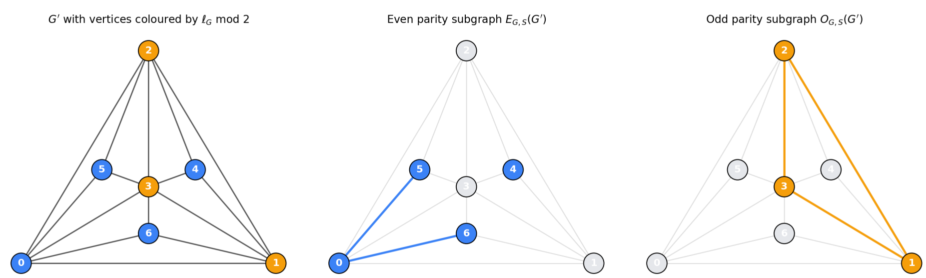


FIGURE 5. Parity subgraphs of $G' = T$ with respect to the level structure of Figure 2 (here we take $G = G' = T$). Left: $T$ with vertices coloured by $\ell_G$ mod 2 (blue = even, orange = odd). Middle: the even parity subgraph $E_{G,S}(G')$, induced on $\{0,4,5,6\}$; only edges with both endpoints even appear. Right: the odd parity subgraph $O_{G,S}(G')$, induced on $\{1,2,3\}$; the highlighted triangle shows that $O_{G,S}(G')$ is not bipartite for this choice of $G'$.

**Definition 2.6** (Facial depth)**.** Let $L_k$ be drawn with the outerplanar embedding inherited from $\Pi_G$, let $D$ be the dual graph of this drawing with the outer face removed, and let $\mathcal{B}$ be the set of inner faces of $L_k$ whose bounding level cycle contains at least one edge of the outer cycle of $L_k$. The *facial depth* of an inner face $F$ of $L_k$ is

$$\mathrm{depth}(F) \;=\; \min_{F' \in \mathcal{B}} \mathrm{dist}_D(F, F'),$$

with the convention $\mathrm{depth}(F) = \infty$ if no such $F'$ exists. An inner face is *isolated* if $\mathrm{depth}(F) \geq 1$.

**Definition 2.7** (Surface switch)**.** A *surface switch* is an edge switch (Definition 2.4) applied to an edge incident to two level cycles, one of facial depth $d$ and the other of facial depth $d-1$.

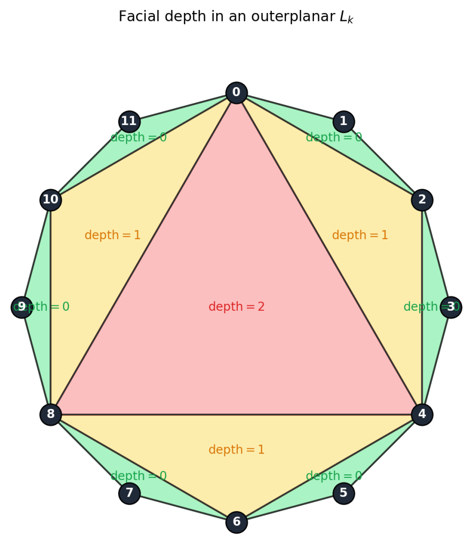


FIGURE 6. Facial depths in a maximal outerplanar graph on 12 vertices. The six green ear-triangles share an edge with the outer 12-cycle and so lie in $\mathcal{B}$ (depth 0). The three yellow "in-between" triangles $(0, 2, 4), (4, 6, 8), (0, 8, 10)$ have only diagonal edges but each is dual-adjacent to ears, giving them depth $= 1$. The central triangle $(0, 4, 8)$ is also all-diagonal; its dual neighbours are the three depth-1 triangles, so it is isolated with depth $= 2$.

**Definition 2.8** (Balanced surface switch)**.** Let $\sigma$ be a surface switch on an edge $e = uv$ separating an inner face $F$ of $L_k$ of depth $d \geq 1$ from an adjacent inner face $F' = uvx$ of depth $d - 1$. We say $\sigma$ is *balanced* if each of the two edges of $\partial F'$ other than $uv$ (namely $ux$ and $vx$) either lies on the outer cycle of $L_k$ or is shared with an inner face of $L_k$ of depth $d - 2$.

When $d = 1$ the condition reduces to "both $ux$ and $vx$ lie on the outer cycle of $L_k$", because no inner face has depth $-1$; in that case $F'$ is a triangular "ear" hanging off $uv$.

## 3. Outerplanarity of level components

For each integer $k \geq 0$ and each $(G, S)$, write $L_k$ for the subgraph of $G$ induced by the level-$k$ vertices. A *level component* of $G$ (with respect to $S$) is a connected component of some $L_k$.

**Theorem 3.1.** *For every plane triangulation $G$ and every level source $S$ of $G$, every level component of $G$ is outerplanar.*

*Proof.* Since every subgraph of an outerplanar graph is outerplanar, it suffices to show that each level subgraph $L_k$ is outerplanar. For $k = 0$, $L_0$ is either a single vertex (when $S$ is a degree-3 vertex) or the triangle bounding the source face (when $S$ is a face), both outerplanar.

Fix $k \geq 1$ and let $D_k$ be the drawing of $L_k$ inherited from $\Pi_G$. Let $F^*$ be the face of $D_k$ containing the source. Suppose for contradiction that some $u \in L_k$ does not lie on $\partial F^*$, so $u$ lies on the boundary of some other face of $D_k$. Take any path $P$ in $G$ from $v_0 \in S$ to $u$. As a curve in $\Pi_G$, $P$ starts in $F^*$ and ends at a point off $\partial F^*$, so it must transition from $F^*$ to a different face of $D_k$; in a planar embedding this can happen only at a vertex of $D_k$, that is, at a level-$k$ vertex $w$ on $P$. Either $w \neq u$ (so $P$ has length $\geq \mathrm{dist}_G(S, w) + 1 \geq k + 1$), or $w = u$ (contradicting $u \notin \partial F^*$). Since every $S$-to-$u$ path has length $\geq k + 1$, $\mathrm{dist}_G(S, u) \geq k + 1$, contradicting $u \in L_k$. □

**Lemma 3.2.** *Let $C$ be a level component of $G$ with respect to $S$, drawn with the outerplanar embedding inherited from $\Pi_G$, and let $D$ be its inner-face dual. If $F$ is an inner face of $C$ with* $\mathrm{depth}(F) = d > 0$*, then $F$ is dual-adjacent to an inner face $F'$ with* $\mathrm{depth}(F') = d - 1$.

*Proof.* By Theorem 3.1, $C$ is outerplanar, so the inner-face dual $D$ is a forest (a standard fact; a tree when $C$ is 2-connected).

Each leaf $F_\ell$ of $D$ contains a single interior edge of $C$, so the remaining edges of $\partial F_\ell$ lie on the outer cycle of $C$. In particular $F_\ell$ has at least one outer-cycle edge, so $F_\ell \in \mathcal{B}$ and $\mathrm{depth}(F_\ell) = 0$. Hence every tree component of $D$ contains an element of $\mathcal{B}$, so the depths of all of its vertices are finite.

Choose a shortest path $F = F_0, F_1, \ldots, F_d = F^*$ in $D$ from $F$ to some $F^* \in \mathcal{B}$ realising $\mathrm{depth}(F) = d$. The suffix $F_1, \ldots, F_d$ witnesses $\mathrm{depth}(F_1) \leq d - 1$. If $\mathrm{depth}(F_1) \leq d - 2$, prepending the edge $F F_1$ to a witnessing path would give $\mathrm{depth}(F) \leq d - 1$, contradicting $\mathrm{depth}(F) = d$. Hence $\mathrm{depth}(F_1) = d - 1$, and we may take $F' := F_1$. □

**Proposition 3.3.** *Let $\sigma$ be a balanced surface switch on the edge $e = uv$ separating $F$ (depth $d \geq 1$) from $F' = uvx$ (depth $d - 1$), and let $G' = G - uv + wx$ be the result of the underlying edge flip, with $uvw, uvx$ the two triangular faces of $G$ at $uv$. Then in $L'_k$ (the level-$k$ subgraph of $G'$, with the level assignment of $G$):*

*(1) the level cycle $\partial F$ is destroyed; and*
*(2) one or two new inner faces appear in $L'_k$, each of depth exactly $d - 1$.*

*Proof.* The flip removes $uv$ from $G$, so $\partial F$ is no longer a cycle of $L'_k$, proving (1). For (2) we split on whether the new edge $wx$ re-enters $L_k$.

**Case (i): $\{w, x\} \not\subseteq L_k$.** Then $L'_k = L_k - uv$. The faces $F$ and $F'$ merge into a single new inner face $\widetilde{F}$ with boundary $(\partial F \cup \partial F') \setminus \{uv\}$. The dual neighbours of $\widetilde{F}$ in $L'_k$ are exactly the former neighbours of $F$ and $F'$ other than each other; in particular they include all inner faces previously adjacent to $F'$ across $ux$ or $vx$, whose depths are at most $d - 2$ by Lemma 3.2 applied to $F'$ (when $d \geq 2$). Thus $\mathrm{depth}(\widetilde{F}) \leq d - 1$.

For the matching lower bound, every neighbour of $\widetilde{F}$ has depth $\geq d - 2$ (neighbours inherited from $F$ have depth $\geq d - 1$; neighbours inherited from $F'$ have depth $\geq d - 2$). When $d \geq 2$, neither $F$ nor $F'$ has an outer-cycle edge, so neither does $\widetilde{F}$, giving $\mathrm{depth}(\widetilde{F}) \geq d - 1$. When $d = 1$, $F' \in \mathcal{B}$ and its outer-cycle edge (necessarily distinct from the interior edge $uv$) survives on $\partial\widetilde{F}$, so $\widetilde{F} \in \mathcal{B}'$ and $\mathrm{depth}(\widetilde{F}) = 0 = d - 1$. In either case $\mathrm{depth}(\widetilde{F}) = d - 1$, giving the unique new face required by (2).

**Case (ii):** $\{w, x\} \subseteq L_k$**.** Then $F = uvw$ and $F' = uvx$ are triangular faces of $L_k$, and $L'_k = L_k - uv + wx$. The chord $wx$ splits the quadrilateral $\partial(F \cup F')$ into two triangular faces $A = uwx$ and $B = vwx$ of $L'_k$. We show $\text{depth}(A) = \text{depth}(B) = d - 1$.

By symmetry it suffices to handle $A$. The dual neighbours of $A$ in $L'_k$ are $A_{uw}$ (the inner face across $uw$, unchanged from $L_k$), $A_{ux}$ (the inner face across $ux$, unchanged), and $B$ (across the new edge $wx$). By balancedness of $\sigma$ applied to the edge $ux$:

- if $ux$ lies on the outer cycle of $L_k$, it remains on the outer cycle of $L'_k$, so $A \in \mathcal{B}'$ and $\text{depth}(A) = 0$ (which equals $d - 1$ because the balanced-with-outer-cycle case forces $d = 1$); or
- if $A_{ux}$ is an inner face, balancedness gives $\text{depth}(A_{ux}) = d-2$, so $\text{depth}(A) \leq 1 + (d - 2) = d - 1$.

For the lower bound in the second sub-case ($d \geq 2$): $A$'s edges are $uw$ (an edge of $F$, interior because $F$ has depth $d \geq 1$), $wx$ (new, not on the outer cycle), and $ux$ (interior in this sub-case), so $A \notin \mathcal{B}'$. Moreover every neighbour of $A$ has depth $\geq d - 2$: $A_{uw}$ inherits depth $\geq d - 1$ from being a former neighbour of $F$, $A_{ux}$ has depth $d - 2$, and $B$ has depth $\geq d - 2$ by the same argument applied symmetrically. Therefore $\text{depth}(A) \geq d - 1$, and combined with the upper bound, $\text{depth}(A) = d - 1$. □

**When does a balanced surface switch exist?** For a chord $uv$ of a maximal outerplanar graph, the *span* of $uv$ is the minimum, over the two arcs from $u$ to $v$ on the outer cycle, of the number of outer-cycle vertices strictly between them.

**Observation 3.4.** For $d = 1$, an inner face $F$ admits a balanced surface switch on some edge iff at least one edge of $F$ has span 1 in the outer cycle of $L_k$. The opposite triangle across that edge – using the single outer-cycle vertex between its endpoints – is then an ear of $F$ in $\mathcal{B}$, satisfying the $d = 1$ form of Definition 2.8.

The smallest maximal-outerplanar configuration violating this is a 9-vertex outer cycle triangulated so that the unique interior face $F = (0, 3, 6)$ has spans $(2, 2, 2)$ on its three edges (Figure 7). Each depth-0 neighbour of $F$ carries exactly one outer-cycle edge, not two, so none qualifies as an ear of $F$; no balanced surface switch is available.

**Preprocessing toward balanced switches.** When $F$ has depth $d \geq 1$ but admits no balanced surface switch, perform a single (unbalanced) surface switch on any edge of $F$ shared with a depth-$(d - 1)$ neighbour. By Proposition 3.3 the result is at least one new depth-$(d - 1)$ face; in Case (ii) it is accompanied by a new depth-$d$ face $A$ that replaces $F$ as the next candidate. The hope is that the resulting $A$ admits a balanced surface switch, or that iterating the preprocessing eventually exposes one.

**Example 3.5.** On the 9-vertex example, the (unbalanced) surface switch on edge $uv = 03$ – with $F' = (0, 2, 3)$, third vertex $x = 2$, and $w = 6$ – flips $03 \mapsto 26$ in $G$ and produces $A = (0, 2, 6)$ at depth 1. The new face has spans $(1, 3, 2)$ on its edges, and the ear $(0, 1, 2)$ across the span-1 edge 02 is now a balanced surface-switch target on $A$ (Figure 8).

We do not have a general termination theorem. The natural candidate monovariant for $d = 1$ is the minimum span among edges of the current depth-1 face that

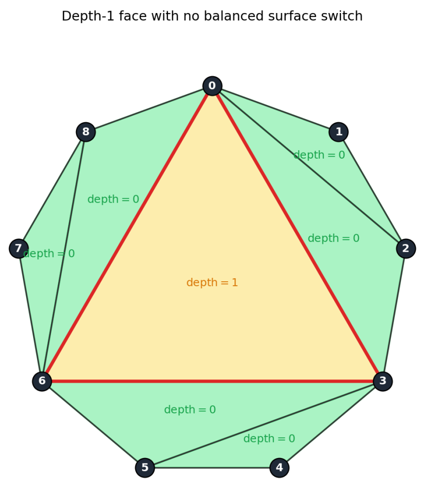


FIGURE 7. 9-vertex maximal outerplanar $L_k$. $F = (0, 3, 6)$ has depth $= 1$ and all three of its edges have span 2, so none of $F$'s depth-0 neighbours is an ear. No balanced surface switch is available on $F$.

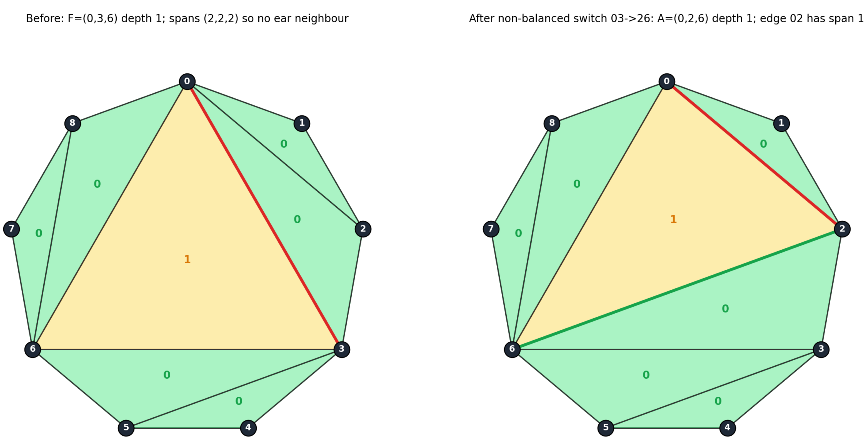


FIGURE 8. One step of preprocessing on the 9-vertex example. Left: $F = (0, 3, 6)$ has no edge of span 1; the chosen surface-switch edge $uv = 03$ (red) is unbalanced. Right: after the switch $03 \mapsto 26$ (green), the new depth-1 face $A = (0, 2, 6)$ has its edge 02 (red) at span 1, exposing the ear $(0, 1, 2)$ as a balanced surface-switch target.

are shared with a depth-0 neighbour; in Example 3.5 this drops from 2 to 1 in a single step. Whether such a monovariant strictly decreases under every unbalanced surface switch – and a corresponding statement for $d \geq 2$, where balancedness depends on depth-$(d-2)$ structure rather than just spans – remains open.

**The $d \geq 2$ analog and recursive lopsidedness.** For $d \geq 2$ the obstruction to a balanced surface switch is no longer "$F$ has no edge of span 1": it is recursive. We say a depth-$(d-1)$ neighbour $F' = uvx$ of $F$ is *lopsided* if exactly one of its non-$F$ neighbours has depth $d-2$ (the other being deeper or an interior face of depth $d-1$). $F$ admits a balanced surface switch iff at least one depth-$(d-1)$ neighbour is not lopsided.

The analog of the 9-vertex example at $d = 2$ is a 21-vertex configuration where the unique depth-2 face $F = (0, 7, 14)$ has three depth-1 neighbours $(0, 3, 7)$, $(7, 10, 14)$, $(14, 17, 0)$, each lopsided: their depth-1 "deep side" is a degree-3 face $(3, 5, 7)$, $(10, 12, 14)$, $(17, 19, 0)$ that itself reaches depth 0 via two ears. So the obstruction at $F$ is one layer of lopsidedness; after a single preprocessing step the new depth-2 face $(3, 7, 14)$ sees the previously-hidden balanced descender as a direct neighbour and the algorithm terminates immediately.

Stacking lopsidedness yields a 24-vertex example (Figure 9) where every depth-1 neighbour of $F$ is lopsided *and* the depth-1 degree-3 face inside each arm $(G_i)$ is itself lopsided. Two preprocessing steps are needed before a balanced switch becomes available: the active depth-2 face migrates from $(0, 8, 16)$ to $(2, 8, 16)$ to $(4, 8, 16)$, at which point the *innermost* depth-1 face $(4, 6, 8)$ – whose two non-$F$ neighbours $(4, 5, 6)$ and $(6, 7, 8)$ are both ears – becomes a direct neighbour and the balanced condition is satisfied. After the balanced switch, 10 further balanced switches drive every face to depth 0.

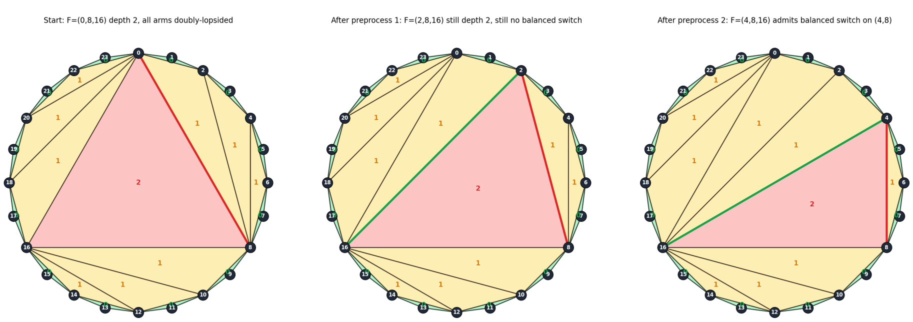


FIGURE 9. Recursive lopsidedness at $d = 2$. Left: $F = (0, 8, 16)$ depth 2, every arm doubly-lopsided. Middle: one preprocessing switch $(0, 8) \mapsto (2, 16)$ exposes the first lopsided layer; the new depth-2 face $(2, 8, 16)$ still has no balanced switch. Right: a second preprocessing switch $(8, 2) \mapsto (4, 16)$ reaches the inner balanced face $K_0 = (4, 6, 8)$, whose two non-$F$ neighbours are both ears; the depth-2 face $(4, 8, 16)$ now admits a balanced surface switch on edge $(4, 8)$.

**Empirical termination.** On every tested configuration, iterated preprocessing terminates and the algorithm

while max-depth face $F$ has depth$(F) > 0$: do a balanced switch if available, else preprocess drives every face to depth 0. The observed step count is

| configuration | $n$ | $d_{\max}$ | total switches |
|---|---|---|---|
| no-balanced $d = 1$ (Figure 7) | 9 | 1 | 4 |
| singly-lopsided $d = 2$ (Figure 9 left only) | 21 | 2 | 8 |
| doubly-lopsided $d = 2$ (Figure 9) | 24 | 2 | 13 |

Each preprocessing step appears to advance the active maximum-depth face one vertex along the lopsided arm of the chosen depth-$(d-1)$ neighbour, peeling off one layer of recursive lopsidedness. The remaining open question is to identify the monovariant that captures this: a candidate is the total number of triples $(F, F', F'')$ where $F' \in N(F)$ is lopsided and $F'' \in N(F')$ is its depth-$d-1$ "deep side". We do not yet have a proof that this strictly decreases under every unbalanced surface switch on a maximum-depth face.

**Question 3.6.** Does iterated preprocessing always reach a balanced surface switch in finitely many steps? Equivalently, is there a monovariant on the inner-face structure of $L_k$ that strictly decreases at every unbalanced surface switch on a maximum-depth face?