DUAL DECOMPOSITION OF MINIMAL COUNTEREXAMPLES

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Abstract.

1. The minimal counterexample

Throughout, a *triangulation* is a simple plane graph, with a fixed embedding, in which every face — including the outer face — is bounded by a triangle. We first reduce to triangulations, then record the degree properties a smallest counterexample must have.

Lemma 1.1 (Reduction to triangulations)**.** *If every triangulation is properly* 4*-vertex-colourable, then so is every plane graph.*

Proof. Let H be a plane graph. Add edges to H, maintaining planarity, until no further edge can be added; the result is a triangulation H^+ on the same vertex set with $E(H) \subseteq E(H^+)$. A proper 4-colouring of H^+ restricts to a proper 4-colouring of H, since every edge of H is an edge of H^+. □

By Lemma 1.1, if the Four Colour Theorem fails then it fails for some triangulation. We may therefore make the following assumption.

Definition 1.2 (Minimal counterexample)**.** Let G be a triangulation on the fewest vertices that admits no proper 4-vertex-colouring. We call G a *minimal counterexample.* By minimality, every triangulation on fewer than $|V(G)|$ vertices is properly 4-colourable.

Remark 1.3. Since every triangulation on at most four vertices is properly 4-colourable (the largest being K_4), a minimal counterexample has $|V(G)| \geq 5$; the degree bound below sharpens this to $|V(G)| \geq 12$.

Lemma 1.4 (Minimum degree)**.** *A minimal counterexample G has minimum degree $\delta(G) \geq 5$.*

Proof. Suppose some vertex v has $\deg(v) = d \leq 4$.

If $d \leq 3$, let $G' = G - v$. Then G' is a plane graph on fewer vertices, so by Definition 1.2 and Lemma 1.1 it has a proper 4-colouring. The at most three neighbours of v use at most three colours, so a fourth colour is free for v, extending the colouring to G — a contradiction.

If $d = 4$, again 4-colour $G - v$. If the four neighbours of v use at most three colours we extend as before, so assume they receive all four colours; let v_1, v_2, v_3, v_4 be the neighbours in cyclic order around v, coloured $1, 2, 3, 4$. Consider the subgraph

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induced by the colour classes 1 and 3, and let K be its connected component containing v_1. If $v_3 \notin K$, swap colours 1 and 3 on K; now no neighbour of v is coloured 1, freeing it for v. If $v_3 \in K$, then a 1–3 Kempe chain joins v_1 to v_3, and this chain together with v encloses exactly one of v_2, v_4; hence the 2–4 component containing v_2 cannot also reach v_4, and swapping colours 2 and 4 on it frees colour 2 for v. Either way the colouring extends to G, a contradiction.

Hence $\delta(G) \geq 5$. □

2. The reduced dual

Write G' for the dual of G: since G is a triangulation, G' is a cubic plane graph in which each vertex of G corresponds to a face of G', each face of G to a vertex of G', and each edge to a dual edge. A vertex of G of degree k corresponds to a k-gonal face of G'.

By Lemma 1.4, $\delta(G) \geq 5$, and Euler's formula gives $\sum_{u \in V(G)}(6 - \deg u) = 12$, so G has a vertex of degree exactly 5 (indeed at least twelve). Fix such a vertex v. Its dual face F_v is a pentagon, bounded by the five dual vertices corresponding to the five faces of G incident to v.

Definition 2.1 (Reduced dual)**.** Let v be a degree-5 vertex of G with pentagonal dual face F_v, and fix an index $i \in \{0, 1, 2, 3, 4\}$. The *reduced dual* $\widehat{G}'_{v,i}$ is the plane graph obtained from G' as follows.

(1) Delete the five dual vertices on the boundary of F_v, together with all edges incident to them. Each deleted vertex is cubic, with two edges on ∂F_v and one edge leaving F_v; deleting the five boundary vertices therefore removes the five external edges as well, dropping their five outer endpoints from degree 3 to degree 2. These five degree-2 vertices lie on the boundary of a single face F of the resulting graph.
(2) List the five degree-2 vertices in clockwise order around F as $A = (A_0, A_1, A_2, A_3, A_4)$.
(3) Add a new vertex v_n and join it to A_i, A_{i+1}, and A_{i+2} (indices mod 5) by three new edges.
(4) Add a new edge between A_{i+3} and A_{i+4} (indices mod 5).

Remark 2.2. Steps (3) and (4) restore cubicity: A_i, A_{i+1}, A_{i+2} each gain one edge to v_n and A_{i+3}, A_{i+4} each gain the new edge, so all five return to degree 3, and v_n has degree 3. Since $A_i, \ldots, A_{i+2}$ and A_{i+3}, A_{i+4} are each consecutive along ∂F, the new vertex and edge can be drawn inside F without crossings, so $\widehat{G}'_{v,i}$ is again a cubic plane graph. The construction depends on the choice of i up to the rotational symmetry of A.

Definition 2.3 (Edges of the reduced dual)**.** The four edges added in steps (3) and (4) of Definition 2.1 are named as follows. The chord $A_{i+3}A_{i+4}$ is the *merged edge*; the edge $A_{i+1}v_n$ is the *spike edge*; and the edges A_iv_n and $A_{i+2}v_n$ are the *side edges*. In the $i = 0$ case of Figure 1 these are $\{A_3, A_4\}$, $\{A_1, v_n\}$, and $\{A_0, v_n\}, \{A_2, v_n\}$ respectively.

We will use the following structural fact about proper 3-edge-colourings near a pentagonal face of a cubic plane graph; it is stated for a generic such graph H, not specifically for the reduced dual.

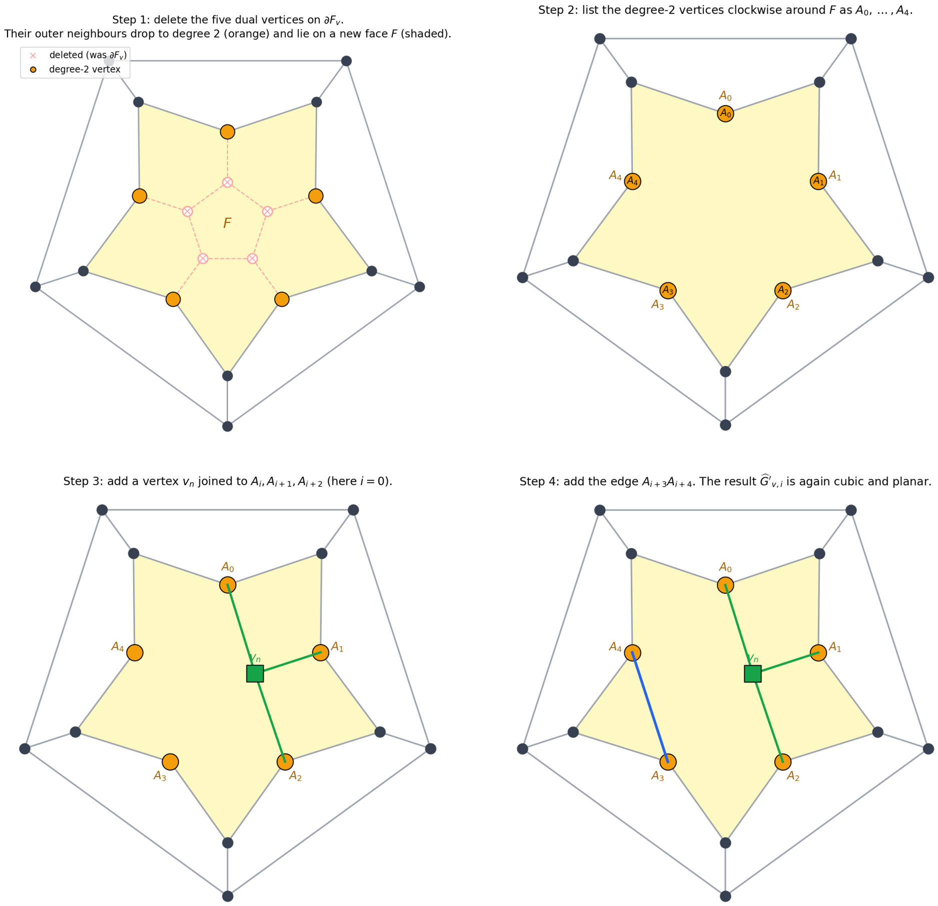

FIGURE 1. The four steps of Definition 2.1, illustrated on G' = the dodecahedron (dual of the icosahedron) with F_v the inner pentagon and $i = 0$. Top left: delete the five boundary vertices of F_v, leaving five degree-2 vertices on a new face F. Top right: order them clockwise as $A_0, \dots, A_4$. Bottom left: add v_n joined to A_0, A_1, A_2. Bottom right: add the chord A_3A_4, giving the cubic plane graph $\widehat{G}'_{v,0}$.

Lemma 2.4 (Pentagonal externals)**.** *Let H be a cubic plane graph and F a pentagonal face of H, with ∂F traversed clockwise as u_0, u_1, u_2, u_3, u_4. For each i let f_i be the unique edge of H incident to u_i that does not lie on ∂F. An assignment φ of colours from $\{1, 2, 3\}$ to the ten edges incident to $\{u_0, \dots, u_4\}$ is proper at every u_i if and only if there is some index j such that*

$$\varphi(f_j) = \varphi(f_{j+1}) = \varphi(f_{j+2}) \quad \text{and} \quad \{\varphi(f_{j+3}), \varphi(f_{j+4})\} = \{1, 2, 3\} \setminus \{\varphi(f_j)\},$$

indices mod 5.

Proof. Write $e_i = u_iu_{i+1}$ for the boundary edges of ∂F (indices mod 5). A colouring φ is proper at every u_i if and only if at each u_i the three incident edges e_{i-1}, e_i, f_i

receive three distinct colours; whenever this holds, $\varphi(f_i)$ is forced to be the unique colour in $\{1,2,3\} \setminus \{\varphi(e_{i-1}), \varphi(e_i)\}$, and φ restricts to a proper 3-edge-colouring of the cycle ∂F.

(⇒) The line graph of ∂F is C_5, whose maximum independent set has size 2, so no colour appears more than twice on ∂F; and since ∂F is an odd cycle, all three colours appear. The colour multiset on $(\varphi(e_0), \ldots, \varphi(e_4))$ is therefore $(2,2,1)$, with the singleton at a unique position. Cyclically shifting indices we may place this position at 0; let c be the singleton colour. The remaining four edges form the path $e_1e_2e_3e_4$, which by propriety alternates between the other two colours, so for some labelling $\{a,b,c\} = \{1,2,3\}$,

$$(\varphi(e_0), \varphi(e_1), \varphi(e_2), \varphi(e_3), \varphi(e_4)) = (c, a, b, a, b).$$

Reading off the forced values of $\varphi(f_i)$,

$$\varphi(f_0) = a, \quad \varphi(f_1) = b, \quad \varphi(f_2) = \varphi(f_3) = \varphi(f_4) = c,$$

which is the lemma's pattern at $j = 2$ (the cyclic shift maps this back to the corresponding j in the original indexing). This case is the unique proper 3-edge-colouring of ∂F up to cyclic shift and permutation of $\{1,2,3\}$ (since $5 \cdot 3! = 30 = P(C_5, 3)$, the chromatic polynomial of C_5 at 3), so it exhausts every proper φ.

(⇐) The lemma's hypothesis is invariant under cyclic shifts of indices and under permutations of $\{1,2,3\}$, so we may assume $j = 2$, $\varphi(f_2) = \varphi(f_3) = \varphi(f_4) = c$, $\varphi(f_0) = a$, and $\varphi(f_1) = b$, with $\{a,b,c\} = \{1,2,3\}$. Propriety at u_i and u_{i+1} requires $\varphi(e_i) \notin \{\varphi(f_i), \varphi(f_{i+1})\}$, which gives

$$\varphi(e_0) = c, \quad \varphi(e_1) = a, \quad \varphi(e_2) \in \{a,b\}, \quad \varphi(e_3) \in \{a,b\}, \quad \varphi(e_4) = b.$$

The remaining propriety condition $\varphi(e_{i-1}) \neq \varphi(e_i)$ holds automatically at u_0, u_1, u_4, forces $\varphi(e_2) = b$ at u_2, and then forces $\varphi(e_3) = a$ at u_3. The resulting triples $(\varphi(e_{i-1}), \varphi(e_i), \varphi(f_i))$ at u_0, u_1, u_2, u_3, u_4 are

$$(b,c,a), \quad (c,a,b), \quad (a,b,c), \quad (b,a,c), \quad (a,b,c),$$

each a permutation of $\{1,2,3\}$, so φ is proper at every u_i. □

Remark 2.5. The two-element condition $\{\varphi(f_{j+3}), \varphi(f_{j+4})\} = \{1,2,3\} \setminus \{\varphi(f_j)\}$ cannot be dropped: a 3-colouring satisfying $\varphi(f_j) = \varphi(f_{j+1}) = \varphi(f_{j+2})$ alone need not extend, e.g. $(1,1,1,1,2)$.

Since $\widehat{G}'_{v,i}$ is the dual of a triangulation on fewer vertices than G, it is 3-edge-colourable by the minimality of G. The following lemma constrains every such colouring.

Lemma 2.6. *Let G be a minimal counterexample, and let $\widehat{G}'_{v,i}$ be a reduced dual of its dual G'. Then in every proper 3-edge-colouring of $\widehat{G}'_{v,i}$, the merged edge and the spike edge receive the same colour.*

Proof. □