

NESTED TIRE DECOMPOSITIONS OF PLANE TRIANGULATIONS

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ABSTRACT. We establish the foundational structure of nested level-induced tire decompositions of a plane triangulation G . A *level source* of G induces a BFS layering of G and endows the inner planar dual G' with a *dual depth* grading. The basic object of study is the *tire graph* T — a plane graph whose outer and inner boundaries bound a closed planar region, the *tire tread* R , triangulated by the *annular edges* E_{ann} . Our main structural results are the *tire-component lemma*, the *tire-tread partition theorem*, and the rooted *tire-tree decomposition*, which together organize the bounded faces of G into nested tire treads.

1. INTRODUCTION

A classical theorem of Tait recasts the Four Colour Theorem in dual, edge-colouring terms: a plane triangulation G is properly 4-vertex-colourable if and only if its dual cubic graph G' is properly 3-edge-colourable. Thus a minimal counterexample to the Four Colour Theorem — a smallest triangulation admitting no proper 4-colouring — corresponds to a smallest cubic plane graph admitting no proper 3-edge-colouring.

The structural study of such a minimal counterexample is the overarching motivation for the present line of work. This first paper establishes the foundational vocabulary — level sources, dual depth, tire graphs, and partial tire duals — on which subsequent papers in the series build. In particular, the companion paper [3] uses these definitions to develop nested-cycle structure theorems and chain-pigeonhole conjectures for tire annular subgraphs of G' .

Related work. The structural object underlying this programme — the set of proper 4-colourings of a boundary cycle that extend to a colouring of a bounded planar region — is classical. Birkhoff’s reducibility analysis of the diamond configuration [4] is the earliest instance of computing such extension sets to attack the Four Colour Theorem; the chromatic polynomial framework of Birkhoff and Lewis [5] systematized the counting. Tutte studied how the chromatic polynomial of a rooted planar triangulation decomposes along its outer boundary [8] and developed an algebraic theory of graph colourings organised around separating subgraphs [7, 6]. The most recent and structurally closest parallel is Dvořák and Lidický’s analysis of *coloring count cones* [11], which characterises the possible boundary-extension functions on a fixed outer cycle of a near-triangulation. The Heesch–Appel–Haken approach [9, 10] also uses boundary-extension reasoning, but case-by-case on a finite unavoidable set of local configurations rather than as part of a global structural induction.

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The tire-tree decomposition introduced here differs from each of these in shape rather than ingredients. Birkhoff, Tutte, and Dvořák–Lidický all study *one* boundary; Heesch and the cleaned-up Appel–Haken proof [10] study a finite collection of local boundaries. The present framework organises the entire triangulation into a hierarchy of annular regions glued along level cycles, and asks whether boundary-extension constraints compose compatibly up the hierarchy. To the authors’ knowledge, no prior work on the Four Colour Theorem has been organised around a global nested-cycle decomposition of this kind.

Throughout, $G = (V, E)$ is a plane maximal planar graph (a triangulation) with a fixed planar embedding Π_G . We write $|V| = n$, so $|E| = 3n - 6$ and G has $2n - 4$ triangular faces.

Definition 1.1 (Level source). A *level source* of G is a set $S \subseteq V$ that is either

- a single vertex $\{v\}$ (a *vertex source*), or
- a set inducing a simple cycle in G — i.e. $G[S]$ is a simple cycle of length ≥ 3 (a *cycle source*).

Definition 1.2 (Levels). Given a level source $S \subseteq V$, the *level* of $v \in V$ is $\ell_G(v) = \text{dist}_G(v, S)$, the graph distance from v to the nearest source vertex. We write $L_d := \{v \in V : \ell_G(v) = d\}$ for the *level- d vertex set*, and abbreviate $L_{<d} := \bigcup_{d' < d} L_{d'}$ and $L_{\geq d} := \bigcup_{d' \geq d} L_{d'}$ (similarly $L_{>d}, L_{\leq d}$).

Definition 1.3 (Dual). The *dual* of G , written G' , is the inner (weak) planar dual of G with respect to the embedding Π_G : it has one vertex d_f for each bounded face f of G , and an edge joining d_f and $d_{f'}$ for each edge of G shared by two bounded faces f and f' . The unbounded outer face contributes no vertex, and edges of G on the outer boundary contribute no dual edge. Since G is a triangulation, each vertex $d_f \in V(G')$ corresponds to a triangular face f of G , and we write $V(f) \subseteq V$ for its three incident vertices.

Definition 1.4 (Dual depth). Given a level source $S \subseteq V$, the *dual depth* of a dual vertex $d_f \in V(G')$ is

$$\delta_G(d_f) = \min_{v \in V(f)} \ell_G(v) = \min_{v \in V(f)} \text{dist}_G(v, S),$$

the smallest level among the three vertices of G bounding the face f .

Definition 1.5 (Depth- d dual subgraph and its components). For $d \geq 0$, the *depth- d dual subgraph* is

$$G'_d := G'[\{d_f \in V(G') : \delta_G(d_f) = d\}],$$

the inner-dual subgraph induced on the dual vertices of dual depth d . For a connected component C' of G'_d we write

$$F_{C'} := \{f : d_f \in V(C')\}, \quad V_{C'} := \bigcup_{f \in F_{C'}} V(f),$$

for its set of faces and the vertices of G bounding them, and $R_{C'} := \bigcup_{f \in F_{C'}} f \subseteq |\Pi_G|$ for the closed planar region these faces cover.

Definition 1.6 (Tire graph). A *tire graph* consists of a plane graph T together with an *outer boundary* $B_{\text{out}} \subseteq T$ and a *connected inner outerplanar graph* $O \subseteq T$ with $V(B_{\text{out}}) \cap V(O) = \emptyset$, where

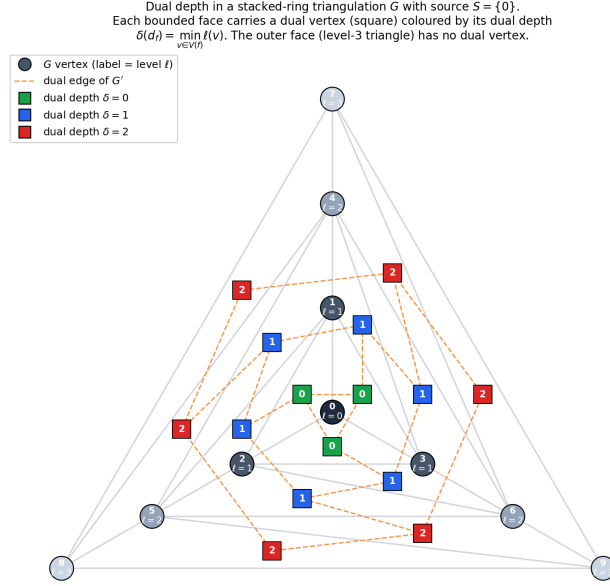


FIGURE 1. Dual depth in a stacked-ring triangulation G with level source $S = \{0\}$. Each G vertex is labelled by its level ℓ . Each bounded face carries a dual vertex (square, joined by dashed dual edges) coloured by its dual depth $\delta(d_f) = \min_{v \in V(f)} \ell(v)$: the central fan has depth 0, the inner annulus depth 1, and the outer annulus depth 2. The outer face (the level-3 triangle) is excluded from the inner dual and carries no dual vertex.

- B_{out} is either a simple cycle of length ≥ 3 or a single vertex (a *degenerate outer boundary*);
- O is a connected outerplanar graph; its *inner boundary* B_{in} is the closed walk in O that traces the boundary of O 's outer face in the inherited embedding, which is a simple cycle when O is 2-connected and a non-simple closed walk in general (visiting bridges twice and cut-vertices multiple times); if $|V(O)| = 1$, we say T has a *degenerate inner boundary*.

At most one of $B_{\text{out}}, B_{\text{in}}$ may be degenerate. The vertex and edge sets of T are

$$V(T) = V(B_{\text{out}}) \cup V(O), \quad E(T) = E(B_{\text{out}}) \cup E(O) \cup E_{\text{ann}},$$

where E_{ann} — the *annular edges* — has the property that, in the plane embedding of T , the closed planar region R bounded externally by B_{out} and internally by B_{in} is partitioned into triangular faces of T whose union is R . We call R the *tire tread* of T and write F_{ann} for this set of triangular faces (the *annular faces*).

When B_{out} is a simple cycle and O is 2-connected, the tread is a closed annulus. More generally, R is a closed planar region that may fail to be a 2-manifold at cut-vertices of O (where two “lobes” of the depth- d region meet at a single vertex); the inner boundary B_{in} is then a non-simple closed walk that visits the cut-vertex multiple times. The relaxed definition accommodates outerplanar inner graphs with

bridges or cut-vertices, while O itself remains connected. When either boundary is degenerate, the tread is a closed disk with that vertex as apex.

We summarize the data of a tire graph as the triple $T = (B_{\text{out}}, O, E_{\text{ann}})$, from which B_{in} , the annular faces F_{ann} , and the tread R are determined; we freely identify a tire graph with its underlying plane graph T .

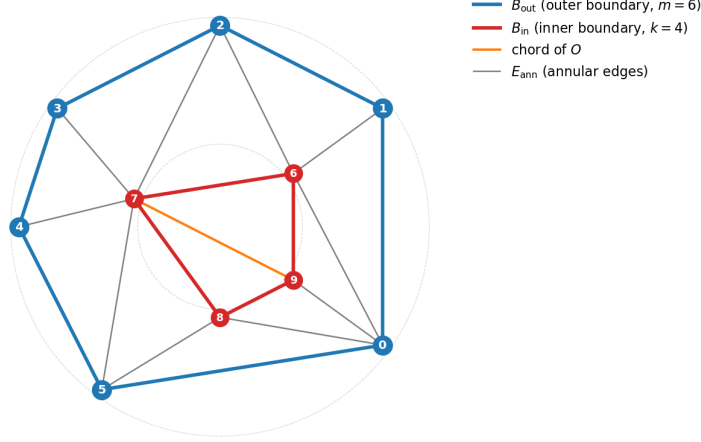


FIGURE 2. A tire graph with non-degenerate boundaries: outer boundary B_{out} a 6-cycle on vertices $0, \dots, 5$ (blue), inner boundary B_{in} a 4-cycle on vertices $6, \dots, 9$ (red), inner outerplanar graph $O = B_{\text{in}} \cup \{7-9\}$ (with one chord, orange), and E_{ann} (grey) tiling the annulus between B_{out} and B_{in} by ten triangular faces.

Definition 1.7 (Medial tire graph). Let $T = (B_{\text{out}}, O, E_{\text{ann}})$ be a tire graph. Let T° be the plane graph obtained from T by deleting every edge of O that is not on the inner boundary walk B_{in} . Equivalently, T° keeps B_{out} , B_{in} , and the annular edges, but omits the chords of the inner outerplanar graph O .

The *medial tire graph* of T , denoted $M_{\text{tire}}(T)$, is obtained from the medial graph $M(T^\circ)$ by deleting every medial edge whose two endpoint-vertices correspond either to two edges of B_{out} or to two edges of B_{in} . Thus $V(M_{\text{tire}}(T))$ is naturally indexed by the non-chord edges of T , and two such vertices are adjacent exactly when the corresponding edges are consecutive on the boundary of a face of T° , except for consecutive pairs lying wholly along one of the two boundary walks. The medial vertices corresponding to edges of E_{ann} are called the *annular medial vertices*.

Remark 1.8. Let $\mu = |V(B_{\text{out}})|$ and $\nu = |V(B_{\text{in}})|$. By Euler's formula on the tire tread R , the tire graph has $\mu + \nu$ triangular faces inside R and $|E_{\text{ann}}| = \mu + \nu$ annular edges when neither boundary is degenerate; when exactly one boundary is degenerate (so $\min(\mu, \nu) = 1$), there are $\mu + \nu - 1$ triangular faces and $|E_{\text{ann}}| = \mu + \nu - 1$.

Proposition 1.9 (Source-side simple-cycle property). *Let G be a maximal planar graph with planar embedding Π_G and single-vertex source v_0 . Let $d \geq 1$, $v \in L_d$,*

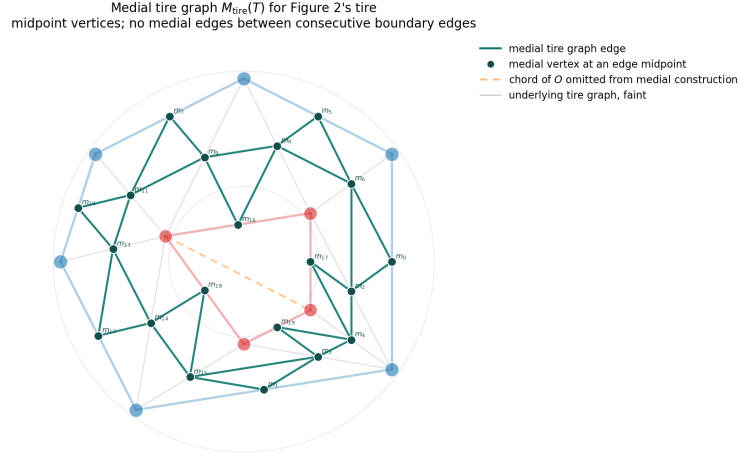


FIGURE 3. The medial tire graph for the tire in Figure 2. The chord of O is drawn faintly and omitted before taking the medial graph; medial edges between consecutive outer-boundary edges or consecutive inner-boundary edges are also omitted. Each medial vertex is placed at the midpoint of its corresponding retained tire edge.

and let C' be a connected component of G'_d such that v is incident to some face in $F_{C'}$. Then the depth- d faces in $F_{C'}$ incident to v form a single contiguous arc in v 's rotation in Π_G .

Equivalently: for any such component, the source-side boundary of $R_{C'}$ is a simple cycle in L_d (no cut-vertices at level d).

Proof. Suppose for contradiction that the depth- d faces in $F_{C'}$ at v lie in two or more disjoint arcs of v 's rotation. Adjacent vertices in G differ in level by at most 1, so a face at v has depth exactly d iff both other vertices have level $\geq d$, and depth $\leq d - 1$ iff at least one has level $d - 1$. Hence the gaps between the depth- d arcs at v are populated by level- $(d - 1)$ neighbours of v , occurring in at least two disjoint arcs of v 's rotation. Pick p in one such gap and q in another.

The BFS ball $G[L_{<d}]$ is connected, so there exists a simple path P in $G[L_{<d}]$ from p to q . Define the closed walk

$$W := v \rightarrow p \rightarrow P \rightarrow q \rightarrow v.$$

Every vertex of P lies in $L_{<d}$, while $\ell(v) = d$, so v is distinct from every vertex of P ; P is simple, so its internal vertices are distinct; and $p \neq q$ since they lie in different gaps. Hence W is a simple cycle in G .

By the Jordan curve theorem, the planar embedding of W divides Π_G into two regions. In v 's rotation, the edges $v - p$ and $v - q$ lie at two specific positions, and they split the rotation into two arcs; each arc lies in one of the two regions determined by W . By choice of p, q , the two arcs of depth- d faces at v in $F_{C'}$ lie in different regions of W (i.e., one arc on each side).

Since C' is connected in G' and contains depth- d faces in both arcs, there is a dual path f_1, f_2, \dots, f_k in G'_d with $f_1, f_k \in F_{C'}$ incident to v in different arcs, and with the intermediate faces f_2, \dots, f_{k-1} not incident to v (a shortest such dual path). Consecutive faces f_i, f_{i+1} share an edge e_i of G ; for $i \geq 2$, both endpoints of e_i lie in $L_{\geq d}$ (since neither f_i nor f_{i+1} is incident to v , all six vertices of these two triangles lie in $L_{\geq d}$). In particular, e_i shares no endpoint with W except possibly v — and v is excluded from f_2, \dots, f_{k-1} .

A planar edge with neither endpoint on a simple closed planar curve W has both of its incident faces on the same side of W . Applying this to each e_i ($i \geq 2$) inductively: starting from f_2 on the same side of W as f_1 (their shared edge $e_1 = w - w'$ opposite to v in f_1 has $w, w' \in L_{\geq d}$ and hence is not on W), the path $f_2 \rightarrow f_3 \rightarrow \dots \rightarrow f_{k-1} \rightarrow f_k$ stays on one side of W .

But f_1 and f_k lie on different sides of W (by construction), contradicting the conclusion that the entire path lies on one side. \square

Lemma 1.10 (Tire-component lemma). *Let G be a maximal planar graph and let $S \subseteq V(G)$ be a level source. Fix a plane embedding Π_G of G in which S lies on the outer face (such an embedding exists for any planar graph and any single-vertex source). For $d \geq 0$, let C' be a connected component of the depth- d dual subgraph G'_d , with faces $F_{C'}$, bounding vertices $V_{C'}$, and region $R_{C'}$ as in Definition 1.5; let $T_{C'} := G[V_{C'}]$ inherit its embedding from Π_G .*

Then $T_{C'}$, with the inherited embedding, is a tire graph in the sense of Definition 1.6. Its outer boundary B_{out} is the side of $R_{C'}$ closer to S in Π_G , namely the level- d subgraph $G[V_{C'} \cap L_d]$ (a simple cycle or single vertex); its connected inner outerplanar graph is $O = G[V_{C'} \cap L_{d+1}]$, and its inner boundary B_{in} is the outer-face boundary closed walk of O in the inherited embedding (a simple cycle when O is 2-connected, a non-simple closed walk in general). The triangular faces of $T_{C'}$ inside the closed boundary region are exactly the faces of G in $F_{C'}$.

Proof. Outerplanarity of the two level parts. By construction S lies on the outer face of Π_G , so the outerplanarity lemma of [2] applies directly with (G, Π_G, S) , giving that $G[L_{d'}]$ is outerplanar for each $d' \geq 0$. Subgraphs of outerplanar graphs are outerplanar, so $G[V_{C'} \cap L_d]$ and $G[V_{C'} \cap L_{d+1}]$ are both outerplanar.

Layer containment. Each $f \in F_{C'}$ has at least one vertex at level d , and adjacent vertices in G differ in level by at most 1; combined with $\delta_G(d_f) = d$, this forces $V(f) \subseteq L_d \cup L_{d+1}$. Hence $V_{C'} \subseteq L_d \cup L_{d+1}$, and $T_{C'}$ has vertex partition $V_{C'} = (V_{C'} \cap L_d) \sqcup (V_{C'} \cap L_{d+1})$.

Boundary edges are monochromatic in level. Each edge e on $\partial R_{C'}$ separates a face $f \in F_{C'}$ from a face $f' \notin F_{C'}$. Because f and f' share the edge e , their dual vertices are adjacent in G' ; if both had depth d they would lie in the same component of G'_d , contradicting $d_f \in C'$ and $d_{f'} \notin C'$. Hence $\delta_G(d_{f'}) \neq d$; combined with the bounded-step property of δ across G' -adjacent faces, $\delta_G(d_{f'}) \in \{d-1, d+1\}$.

- If $\delta_G(d_{f'}) = d-1$, the third vertex w of $f' = \{u, v, w\}$ (where u, v are the endpoints of e) has $\ell(w) = d-1$. Each of u, v has $\ell \in \{d, d+1\}$ (from $V(f) \subseteq L_d \cup L_{d+1}$) and is adjacent to w , forcing $\ell(u), \ell(v) \in \{d-2, d-1, d\} \cap \{d, d+1\} = \{d\}$.
- If $\delta_G(d_{f'}) = d+1$, then all three vertices of f' lie in $L_{\geq d+1}$, so in particular $\ell(u) = \ell(v) = d+1$.

Each connected boundary component thus carries a single type at every edge: any vertex on a boundary component has two boundary edges incident to it (by R1, see below), both of the same type, so its level is fixed. Therefore each boundary component of $\partial R_{C'}$ is monochromatic in level.

Boundary structure. Each connected component of $\partial R_{C'}$ traces a closed walk in G that, by the monochromaticity above, lies entirely in L_d or entirely in L_{d+1} . By Proposition 1.9, the depth- d faces of $F_{C'}$ at any $v \in L_d \cap V_{C'}$ form a single contiguous arc in v 's rotation, so the source-side boundary walk visits each L_d -vertex of $V_{C'}$ exactly once: it is a simple cycle. At vertices $v \in L_{d+1} \cap V_{C'}$ the depth- d faces may split into multiple arcs of v 's rotation; this corresponds exactly to v being a cut-vertex of O , and the inner-side boundary walk visits v correspondingly many times — which is already accommodated by Definition 1.6 (where B_{in} is the outer-face boundary closed walk of O , not necessarily a simple cycle).

Outer boundary. Because S lies on the outer face of Π_G , the boundary curve(s) of $R_{C'}$ on the L_d side are closer to S in the embedding. In the inherited embedding of $T_{C'}$, the unique unbounded face is the merged region containing the rest of Π_G outside $R_{C'}$ on the S side, so its boundary — a simple cycle on L_d (or a single vertex when $V_{C'} \cap L_d = \{v_0\}$, the $d = 0$ case) — serves as B_{out} . We set $B_{\text{out}} := G[V_{C'} \cap L_d]$ if this is a cycle, and the single vertex $\{v_0\}$ in the degenerate case.

Inner outerplanar graph. By the outerplanarity lemma of [2], $G[V_{C'} \cap L_{d+1}]$ is outerplanar. We set $O := G[V_{C'} \cap L_{d+1}]$. The boundary curve(s) of $R_{C'}$ on the L_{d+1} side are exactly the boundary of O 's outer face in the inherited embedding; this outer-face boundary is a single closed walk that traces around O from the outside, traversing any bridge edge twice and visiting cut-vertices multiple times. This walk is the inner boundary B_{in} . No further restriction on O 's internal structure is needed: when the inner side has several lobes meeting through cut-vertices or bridges of O , the outer-face boundary closed walk of the connected graph O captures them by revisiting those vertices or traversing those bridges twice.

Tire structure. The triangular faces of $T_{C'}$ inside the closed boundary region are by construction the depth- d faces in $F_{C'}$, and the edges of $T_{C'}$ are $E(B_{\text{out}}) \cup E(O) \cup E_{\text{ann}}$ where E_{ann} are the edges of G between $V_{C'} \cap L_d$ and $V_{C'} \cap L_{d+1}$ that bound a face of $F_{C'}$. \square

Theorem 1.11 (Tire treads partition the bounded faces). *Let G be a maximal planar graph with planar embedding Π_G and let $S \subseteq V(G)$ be a level source lying on the outer face. For each $d \geq 0$ and each connected component C' of G'_d , let $T^{(d,C')}$ denote the tire graph supplied by Lemma 1.10, with tire tread $R_{C'} \subseteq |\Pi_G|$. Then the collection of treads*

$$\mathcal{R}(G, S) := \{ R_{C'} : d \geq 0, C' \text{ a connected component of } G'_d \}$$

partitions the bounded part of $|\Pi_G|$:

- (i) *every bounded face f of G is contained in exactly one tread $R_{C'} \in \mathcal{R}(G, S)$;*
- (ii) *distinct treads in $\mathcal{R}(G, S)$ have disjoint interiors and may share only boundary edges or vertices.*

Proof. Existence and uniqueness. Each bounded face $f \in F(G)$ has a uniquely-defined dual depth $\delta_G(d_f) \in \mathbb{Z}_{\geq 0}$, so the dual vertex d_f lies in G'_d for $d = \delta_G(d_f)$ and in no other $G'_{d'}$. Within G'_d , the vertex d_f belongs to exactly one connected component C' . By Lemma 1.10, $F_{C'}$ is precisely the set of faces $f' \in F(G)$ with $d_{f'} \in V(C')$; in particular $f \in F_{C'}$, hence $f \subseteq R_{C'}$.

For any other tread $R_{C''} \in \mathcal{R}(G, S)$, the component C'' is either at a different depth $d' \neq d$ (in which case $F_{C''}$ consists of depth- d' faces and $f \notin F_{C''}$) or at depth d but a different component $C'' \neq C'$ (in which case the two components are vertex-disjoint in G'_d , so again $f \notin F_{C''}$). In both cases $f \notin R_{C''}$ (more precisely, f is not one of the triangular faces of G in $F_{C''}$, so f 's interior is not contained in $R_{C''}$).

Disjoint interiors. Each tread $R_{C'}$ is the union of its triangular faces $F_{C'} \subseteq F(G)$; distinct treads correspond to disjoint $F_{C'}$ (by the argument above), and the interiors of distinct G -faces are disjoint. Hence interiors of distinct treads are disjoint.

Coverage. Conversely, every bounded $f \in F(G)$ has $d_f \in V(G')$ with some dual depth d , and thus lies in $R_{C'}$ where C' is its component of G'_d . So $\bigcup_{R \in \mathcal{R}(G, S)} R$ contains every bounded face of G . \square

Remark 1.12. Either boundary part of $T_{C'}$ in Lemma 1.10 may be degenerate. At $d = 0$ with single-vertex source $S = \{v_0\}$ the unique component of G'_0 has $V_{C'} \cap L_0 = \{v_0\}$ as the degenerate *outer* boundary and $V_{C'} \cap L_1$ a cycle (the link of v_0 in G) as the inner boundary. Symmetrically, at $d = D_{\max}$, $V_{C'} \cap L_{D_{\max}+1} = \emptyset$ degenerates to a single deepest vertex serving as the *inner* boundary, with the level- D_{\max} cycle as the outer boundary.

Remark 1.13. Two structural features of $R_{C'}$ that might at first appear to obstruct the tire-graph conclusion are both already accommodated by Definition 1.6:

Cut-vertices of O . A vertex $v \in V_{C'} \cap L_{d+1}$ may have the faces of $F_{C'}$ incident to it split into two or more arcs in v 's rotation in Π_G , separated by faces of higher depth. This corresponds exactly to v being a cut-vertex of $O = G[V_{C'} \cap L_{d+1}]$, and the inner boundary closed walk B_{in} then visits v multiple times — once for each arc. No additional hypothesis is needed.

Non-2-connected inner topology. Even when the inner side of $R_{C'}$ has several lobes joined at cut-vertices or across bridges, the connected inner outerplanar graph O captures this structure as a non-2-connected outerplanar graph, and its outer-face boundary closed walk serves as B_{in} by traversing bridges twice and visiting cut-vertices multiple times.

In the special case $d = 0$ with single-vertex source $S = \{v_0\}$, $R_{C'}$ is the star of v_0 , a topological closed disk with one boundary cycle (the link of v_0); the corresponding tire graph has degenerate outer boundary $\{v_0\}$.

Theorem 1.14 (Inner dual of a tire tread is outerplanar). *Let $T = (B_{\text{out}}, O, E_{\text{ann}})$ be a tire graph, and let Γ be the graph on vertex set $\{d_f : f \in F_{\text{ann}}\}$ with an edge $d_f d_{f'}$ for each interior annular edge of T (= each edge of T whose two incident faces both lie in F_{ann}). Equivalently, Γ is the subgraph induced on F_{ann} of the full tire dual $D(T)$ — the dual of T taken over all of its triangular faces, in which each boundary edge of R contributes a degree-1 vertex. Then Γ is outerplanar.*

Moreover, Γ admits a planar embedding as a (possibly non-simple) Hamilton walk through every d_f , plus zero or more non-crossing chords.

Proof. We argue by cases on whether the tire tread R is a disk or an annulus.

Case 1: R is a closed disk (one of $B_{\text{out}}, B_{\text{in}}$ degenerate, by Definition 1.6). Let v_0 be the degenerate-boundary vertex (the apex) and let $k = |B_{\text{non-deg}}|$ be the length of the non-degenerate boundary cycle. The triangulation of R is a *fan* of k

triangles around v_0 : each triangle has the form $\{v_0, u_i, u_{i+1}\}$ where u_1, \dots, u_k are the boundary-cycle vertices in cyclic order. Each triangle has two spoke edges (= the two edges incident to v_0 , shared with the two neighbouring fan triangles) and one boundary edge (in $B_{\text{non-deg}}$, contributing a leaf in $D(T)$ but no edge in Γ). Hence every d_f has Γ -degree exactly 2, and Γ is a single cycle of length k . Cycles are outerplanar.

See Figure 4 for the disk case ($k = 6$).

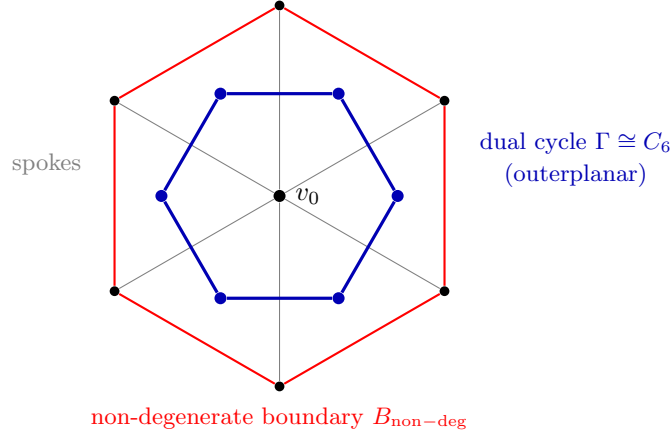


FIGURE 4. Case 1 ($R = \text{disk}$, $k = 6$). The apex v_0 sits at the centre; the non-degenerate boundary $B_{\text{non-deg}}$ (red) is the hexagonal outer cycle; spokes (grey) triangulate the disk into a fan of 6 triangles around v_0 . Each triangle has two spoke edges (interior, contributing Γ -edges) and one boundary edge (contributing a leaf in $D(T)$, no Γ -edge). The inner dual Γ (blue) is the cycle C_6 formed by the six annular face centroids, a manifestly outerplanar graph.

Case 2: R is an annulus (both B_{out} and B_{in} non-degenerate). We construct an explicit outerplanar embedding of Γ as a Hamilton walk plus non-crossing chords.

Step 1: Cyclic ordering of F_{ann} . The boundary of the annular tread is the disjoint union $\partial R = B_{\text{out}} \sqcup \overline{B_{\text{in}}}$ (viewing B_{in} as a closed walk traced in the appropriate orientation). Each boundary edge of R is incident to exactly one annular face: walking around B_{out} in cyclic order produces a sequence $f_1^{\text{out}}, f_2^{\text{out}}, \dots, f_\mu^{\text{out}}$ of (not necessarily distinct) annular faces, one per B_{out} -edge; similarly walking around B_{in} produces a sequence $f_1^{\text{in}}, \dots, f_{\nu_\partial}^{\text{in}}$ where ν_∂ is the length of the inner-boundary walk. Pick any spoke $e^* = uw \in E_{\text{ann}}$ with $u \in V(B_{\text{out}})$ and $w \in V(B_{\text{in}})$; cut R along e^* . This converts the annulus into a closed disk \tilde{R} whose boundary walks once around B_{out} , once along e^* , once around B_{in} in reverse, and once back along e^* . Concatenating the two boundary sequences (in the order dictated by this disk traversal) yields a single cyclic sequence

$$\mathcal{S} = (f_1^{\text{out}}, \dots, f_\mu^{\text{out}}, f_1^{\text{in}}, \dots, f_{\nu_\partial}^{\text{in}})$$

of annular faces with multiplicities.

Step 2: The Hamilton walk. Consecutive entries of \mathcal{S} correspond either to the same annular face (when two adjacent boundary edges meet at a vertex incident to a single annular face) or to two annular faces sharing an interior edge of E_{ann} . In the former case the walk stays at one Γ -vertex; in the latter it uses one Γ -edge. The resulting closed walk in Γ visits every face that appears in \mathcal{S} at least once.

If every $f \in F_{\text{ann}}$ appears in \mathcal{S} (i.e. every annular face has at least one boundary edge of R), the walk is a Hamilton walk in Γ , and we are done up to Step 3. Each annular face with two boundary edges contributes a vertex visited twice; each with three contributes a vertex visited three times.

If some $f \in F_{\text{ann}}$ does not appear in \mathcal{S} (i.e. has no boundary edge of R), then all three edges of f are interior annular edges, so d_f has degree 3 in Γ . Such a face is “trapped” in the interior of the dual graph and appears as the endpoint of a chord. Extend the walk by: whenever it crosses an interior annular edge e shared with a boundary-free face f , detour through f and back. After finitely many such detours (one per boundary-free face), the walk becomes a Hamilton walk visiting every d_f .

Step 3: Non-crossing chords. The Γ -edges not used by the Hamilton walk constructed in Step 2 are the remaining interior annular edges. Each such edge $e \in E_{\text{ann}}$ corresponds to a chord between two non-adjacent positions of \mathcal{S} . In the inherited planar embedding of Γ in R , these chords are drawn as straight segments between annular triangle centroids; *they do not cross* because the underlying E_{ann} edges they cross are themselves non-crossing in the planar embedding of T .

Step 4: Outerplanar embedding. We now lay out Γ as follows: place the $|F_{\text{ann}}|$ vertices on a circle in the cyclic order given by \mathcal{S} (treating multiply-visited faces as single circle vertices). Connect consecutive vertices on the circle by the Hamilton-walk edges, which forms the closed walk. Draw the remaining edges as chords inside the circle. Because the chords were non-crossing in T ’s planar embedding, they remain non-crossing here. All vertices lie on the outer face (the unbounded region outside the circle), making Γ outerplanar. \square

Remark 1.15. In the *spoke-only* case (Definition 1.6 with O 2-connected and E_{ann} consisting only of spokes), every annular face has exactly one boundary edge, every d_f has Γ -degree 2, and the construction of the Theorem 1.14 proof reduces to the classical Hamilton cycle $\Gamma \cong C_{\mu+\nu}$ with zero chords.

Remark 1.16. When O has a bridge $e_{\text{br}} \in E(O)$ whose two incident faces are annular triangles, e_{br} contributes an interior annular edge in Γ rather than two leaves in $D(T)$ (see Definition 1.7 of [3]). The two bridge-incident annular triangles have Γ -degree 3; the resulting Γ has the structure of a Hamilton cycle of length $\mu + \nu_{\partial}$ plus a single chord (length 1). This corresponds to the theta graph $\Theta(1, b, c)$ identified empirically in [3], which has no $K_{2,3}$ subdivision (since one of the three paths has length 1 and so contributes no degree-2 branch vertex), hence is outerplanar as predicted.

Theorem 1.17 (Tire treads form a rooted tree under face containment). *Let G be a maximal planar graph with planar embedding Π_G and let $S \subseteq V(G)$ be a single-vertex level source $\{v_0\}$ lying on the outer face of Π_G . The collection $\mathcal{R}(G, S)$ of tire treads (Theorem 1.11) carries a canonical rooted tree structure $\mathcal{T}(G, S)$ defined as follows.*

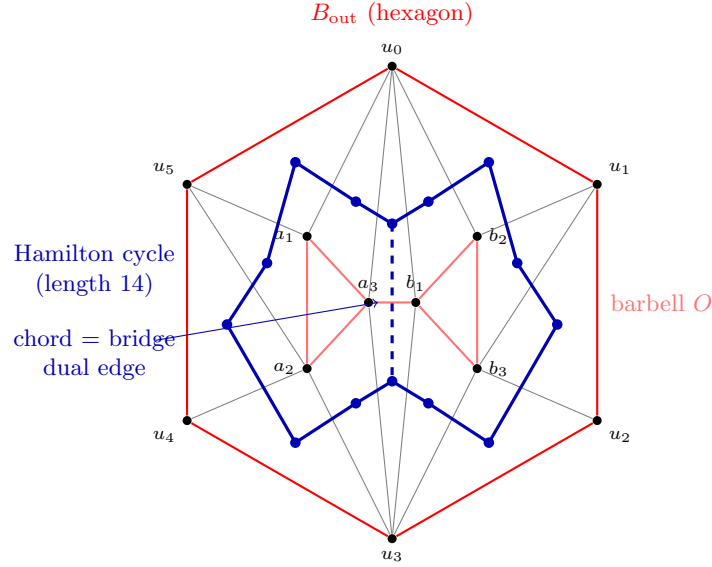


FIGURE 5. Case 2 ($R = \text{annulus}$) with O a barbell. B_{out} is the outer hexagon (red); O has two triangles $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$ joined by the bridge a_3-b_1 (all light red). The annulus is triangulated by 14 annular triangles: 6 “outer-cap” triangles (one per outer edge), 6 “inner-cap” triangles (one per non-bridge edge of O), and 2 “bridge-cap” triangles $\{u_0, a_3, b_1\}$ and $\{u_3, a_3, b_1\}$ adjacent to the bridge. Each blue dot sits at the centroid of an annular triangle; blue edges connect dual vertices whose triangles share an interior annular edge (spoke or bridge). The two bridge-cap vertices have Γ -degree 3 (their triangles have no boundary edge) and are joined by the dashed blue *chord* corresponding to the bridge; the remaining 13 edges form the Hamilton cycle that wraps around the annulus. All 14 vertices lie on the outer face of the cycle-with-chord embedding, so $\Gamma \cong \Theta(1, 7, 7)$ is outerplanar.

- **Root.** The depth-0 tire tread T_0 — the unique tire produced by Lemma 1.10 at $d = 0$, with degenerate outer boundary $B_{\text{out}} = \{v_0\}$ and inner outerplanar graph $O^{(T_0)} = G[L_1]$ — is the root.
- **Parent.** For each tire tread T_c at depth $d \geq 1$, its outer boundary $B_{\text{out}}^{(T_c)}$ is a cycle in L_d . The parent of T_c is the unique tire tread T_p at depth $d - 1$ whose inner outerplanar graph $O^{(T_p)}$ has $B_{\text{out}}^{(T_c)}$ as the boundary cycle of one of its bounded faces. Equivalently, R_c lies inside this bounded face of $O^{(T_p)}$ (which is itself the region of the plane cut off by $B_{\text{out}}^{(T_c)}$ on the side away from S).
- **Children.** The children of a tire tread T_p are in bijection with those bounded faces of $O^{(T_p)}$ whose interiors contain at least one vertex of G at level $\geq d+2$ — equivalently, with the connected components of G'_{d+1} whose tires have outer boundary cycle equal to a bounded face of $O^{(T_p)}$.

Every tire tread except T_0 has exactly one parent; a tire tread may have zero, one, or several children.

Proof. Root is well-defined. At $d = 0$ with single-vertex source $S = \{v_0\}$, the dual subgraph G'_0 is connected (every face of G incident to v_0 has dual depth 0, and they form a single fan around v_0). By Lemma 1.10, the unique component of G'_0 gives the depth-0 tire T_0 described above.

Existence of parent. Fix a tire tread T_c at depth $d \geq 1$ arising from a connected component C'_c of G'_d . Its outer boundary $B_{\text{out}}^{(T_c)} = G[V_{C'_c} \cap L_d]$ is a simple cycle in L_d (Lemma 1.10; the source-side boundary of a tire is always a simple cycle, by Proposition 1.9). The faces of G immediately outside $B_{\text{out}}^{(T_c)}$ on the side facing S have depth $d - 1$ (one of their three vertices lies in L_{d-1} , two in L_d). Let C'_p be the connected component of G'_{d-1} containing the dual vertex of any such face.

Uniqueness of parent. $B_{\text{out}}^{(T_c)}$ is a single simple cycle in G , with a well-defined “ S -side” (the side of the cycle closer to v_0 in Π_G). The depth- $(d - 1)$ faces lying on this side form a single contiguous arc around $B_{\text{out}}^{(T_c)}$ in the dual — they are all G' -adjacent in sequence (each pair of consecutive arc faces shares an edge in $B_{\text{out}}^{(T_c)}$). Hence they all lie in the same connected component C'_p of G'_{d-1} , which is therefore unique.

$B_{\text{out}}^{(T_c)}$ bounds a face of $O^{(T_p)}$. The parent tire T_p has $V(O^{(T_p)}) = V_{C'_p} \cap L_d \supseteq V(B_{\text{out}}^{(T_c)})$. The cycle $B_{\text{out}}^{(T_c)}$ is a subgraph of $O^{(T_p)}$ that bounds a face of $O^{(T_p)}$ in the inherited embedding: the cycle traces around a depth- $\geq d + 1$ region (containing R_c and any descendants of T_c), which is exactly a bounded face of $O^{(T_p)}$.

Children description. The bounded faces of $O^{(T_p)}$ are in bijection with the connected components of G'_d whose faces lie inside those bounded regions (= one component per bounded face, by an argument analogous to the existence-and- uniqueness step above, applied one level deeper).

Tree property. Every non-root T_c has a unique parent at strictly smaller depth. Iterating the parent map strictly decreases depth, terminating at T_0 . No cycles can form (depth is monotone). Hence $\mathcal{T}(G, S)$ is a rooted tree. \square

Remark 1.18. A parent tire T_p has multiple children precisely when its inner outerplanar graph $O^{(T_p)}$ has multiple bounded faces with non-trivial interiors (= containing depth- $\geq d + 2$ vertices of G). This happens, for instance, when $O^{(T_p)}$ has chords or cut-vertices that subdivide its inner region. By contrast, if $O^{(T_p)}$ is a simple cycle (the spoke-only case of Remark 1.15) with a non-empty interior, T_p has exactly one child.

Theorem 1.19 (Tire-tree decomposition). *Let G be a maximal planar graph with planar embedding Π_G and let $v_0 \in V(G)$. The tree of tire treads $\mathcal{T}(G, \{v_0\})$ of Theorem 1.17 decomposes G into nested tires: it is a finite rooted tree, rooted at the depth-0 tread containing v_0 , whose nodes (tire treads) partition the bounded faces of G (Theorem 1.11).*

This decomposition is moreover self-similar. For any tread T in $\mathcal{T}(G, \{v_0\})$ at depth $d \geq 1$, with outer-boundary cycle $C_T := B_{\text{out}}^{(T)}$, let G_T be the sub-graph of G induced by C_T together with all vertices of G lying in the closed planar region $R_T \subset |\Pi_G|$ bounded by C_T on the side of C_T away from v_0 . Then:

- (D1) G_T , with the embedding inherited from Π_G , is a triangulated disk: every bounded face is a triangle, and the outer face is bounded by C_T .
- (D2) Taking C_T as a cycle source of G_T (so C_T has level 0 in G_T and the BFS-from- C_T levels in G_T equal $\ell_G(\cdot) - d$ on $V(G_T)$), the construction of Theorem 1.17 extends to give a rooted tree of tire treads $\mathcal{T}(G_T, C_T)$ whose depth-0 root tread has $B_{\text{out}} = C_T$ and inner outerplanar graph $O = O^{(T)}$.
- (D3) $\mathcal{T}(G_T, C_T)$ is canonically iso to the sub-tree of $\mathcal{T}(G, \{v_0\})$ rooted at T , preserving outer-boundary cycles, inner outerplanar graphs, and the parent-child face correspondence.

In short: pick any vertex $v_0 \in V(G)$ to root the global tree $\mathcal{T}(G, \{v_0\})$ describing the whole graph; pick any tread T in this tree; then T is itself the root of a local tree $\mathcal{T}(G_T, C_T)$ describing the triangulated disk of G inside C_T , with C_T as cycle source. Maximal planar graphs decompose into nested trees of tire treads.

Proof. Decomposition. Theorem 1.17 gives the rooted tree structure of $\mathcal{T}(G, \{v_0\})$, with root the depth-0 tread containing v_0 ; Theorem 1.11 gives that its tire treads partition the bounded faces of G . Finiteness of the tree is immediate from finiteness of G .

(D1) G_T is a triangulated disk. By Lemma 1.10 applied to the component of G'_d that gives rise to T , the outer boundary $C_T = B_{\text{out}}^{(T)}$ is a simple cycle in L_d^G . By the Jordan curve theorem, C_T separates $|\Pi_G| \setminus C_T$ into two open regions; R_T is the closure of the one not containing v_0 . The bounded faces of G_T in its inherited embedding are exactly the bounded faces of G contained in R_T , each of which is a triangle since G is a triangulation. The unbounded face of G_T 's embedding is the complement of R_T , whose boundary is C_T .

(D2) *Level shift.* We show $\text{dist}_{G_T}(v, C_T) = \ell_G(v) - d$ for every $v \in V(G_T)$. When $v \in C_T$ both sides equal 0, so fix $v \in V(G_T) \setminus C_T$.

Step 1: $\text{dist}_G(v, C_T) = \ell_G(v) - d$. A shortest G -path from v to v_0 must visit C_T , since v and v_0 lie in different open regions of $|\Pi_G| \setminus C_T$; let w be its first C_T -vertex. The v -to- w sub-path has length $\geq \text{dist}_G(v, C_T)$ and the w -to- v_0 sub-path has length $\ell_G(w) = d$, so $\ell_G(v) \geq \text{dist}_G(v, C_T) + d$. Conversely, concatenating a shortest G -path from v to a nearest C_T -vertex w' with a shortest G -path from w' to v_0 gives a v -to- v_0 path of length $\text{dist}_G(v, C_T) + d$, so $\ell_G(v) \leq \text{dist}_G(v, C_T) + d$.

Step 2: $\text{dist}_{G_T}(v, C_T) = \text{dist}_G(v, C_T)$. The inequality \geq is automatic since $G_T \subseteq G$. For \leq , pick a shortest G -path π from v to C_T ; we may assume π has no internal vertex in C_T (truncate otherwise). Any internal vertex of π then lies in the same open region of $|\Pi_G| \setminus C_T$ as v , i.e. in $R_T \setminus C_T \subseteq V(G_T)$; every edge of π has both endpoints in $V(G_T)$ and so lies in $E(G_T)$. Hence π is a path in G_T realising $\text{dist}_G(v, C_T)$.

Combining the two steps yields $\text{dist}_{G_T}(v, C_T) = \ell_G(v) - d$, as claimed.

(D3) *Tree iso.* By (D2), $L_k^{G_T} = L_{d+k}^G \cap V(G_T)$ for every $k \geq 0$. For a bounded face f of G_T , dual depth in G_T equals $\min_{u \in V(f)} \ell_{G_T}(u) = \min_{u \in V(f)} \ell_G(u) - d = \delta_G(d_f) - d$. Hence the inner-dual subgraph $(G_T)'_k$ at depth k in G_T is the induced subgraph of G'_{d+k} on the faces of G lying in R_T , and two such faces are dual-adjacent in G'_T iff they are dual-adjacent in G' (the shared edge is in $E(G_T)$).

Step 3: components of $(G_T)'_k$ are precisely the depth- $(d+k)$ descendants of T in $\mathcal{T}(G, \{v_0\})$. We show by induction on k that a component C' of G'_{d+k} has $F_{C'} \subseteq R_T$ iff C' is a depth- $(d+k)$ descendant of T .

For $k = 0$: the components of G'_d are the depth- d treads; the component giving rise to T has its faces in T 's tread region $R \subseteq R_T$, while any other depth- d tread T'' has $C_{T''}$ disjoint from C_T and lying in a different bounded face of $O^{(T'')}$ at depth $d - 1$, hence $R_{T''} \cap R_T = \emptyset$.

For $k \geq 1$: by Theorem 1.17, each component C' of G'_{d+k} has a unique parent C'_p at depth $d + k - 1$, with $B_{\text{out}}^{(C')}$ bounding a face of $O^{(C'_p)}$; equivalently $R_{C'}$ lies inside that bounded face, hence inside $R_{C'_p}$. By the induction hypothesis $R_{C'_p} \subseteq R_T$ iff C'_p is a descendant of T at depth $d + k - 1$, and $R_{C'} \subseteq R_{C'_p}$, so $R_{C'} \subseteq R_T$ iff C' is a descendant of T at depth $d + k$.

Step 4: tread data and child-face correspondence. The Tire-component lemma (Lemma 1.10) and the source-side simple-cycle property (Proposition 1.9) extend verbatim to the cycle-sourced triangulated disk (G_T, C_T) : the proofs use only the triangular structure of bounded faces, the local arrangement of faces around each vertex's rotation, and the connectivity of the BFS ball $G_T[L_{<k}^{G_T}]$ (which holds for every $k \geq 1$ since $L_0^{G_T} = V(C_T)$ is connected as a cycle and each higher level is BFS-adjacent to the previous). Applied to each component of $(G_T)'_k$, the lemma produces a tire graph with outer boundary B_{out} , inner outerplanar graph O , and tread region R identical to those produced by the corresponding component of G'_{d+k} in $\mathcal{T}(G, \{v_0\})$, since these data depend only on level- $d+k$ and level- $(d+k+1)$ vertices and the bounded faces in between — all of which are unchanged when restricting to G_T .

The depth-0 case ($k = 0$) gives a single component, namely the one producing T , with root tread $B_{\text{out}} = C_T$ and $O = O^{(T)}$.

The parent-child face correspondence of Theorem 1.17 is preserved: for any tread T' in $\mathcal{T}(G_T, C_T)$ at depth k , its children correspond to non-trivial bounded faces of $O^{(T')}$, and the bounded faces of $O^{(T')}$ together with the descendant-side interior of each are identical in G_T and in G .

Combining Steps 3 and 4: the bijection $C' \leftrightarrow C'$ (component of $(G_T)'_k$ to corresponding component of G'_{d+k} inside R_T) lifts to a rooted-tree iso $\mathcal{T}(G_T, C_T) \rightarrow$ sub-tree of $\mathcal{T}(G, \{v_0\})$ rooted at T , preserving outer boundaries, inner outerplanar graphs, and the parent-child face correspondence. \square

Remark 1.20. Combining Theorem 1.11 (treads partition the bounded faces of G) with Theorem 1.17 (treads form a rooted tree), any proper coloring problem on G 's bounded faces factors through:

- local coloring problems on each tread (the inner dual of each tread is outerplanar by Theorem 1.14), plus
- consistency constraints along parent-child interfaces (the cycle $B_{\text{out}}^{(T_c)}$ shared between a child and the face of its parent's $O^{(T_p)}$).

This is the structural setup underlying the chain-pigeonhole program for tire treads.

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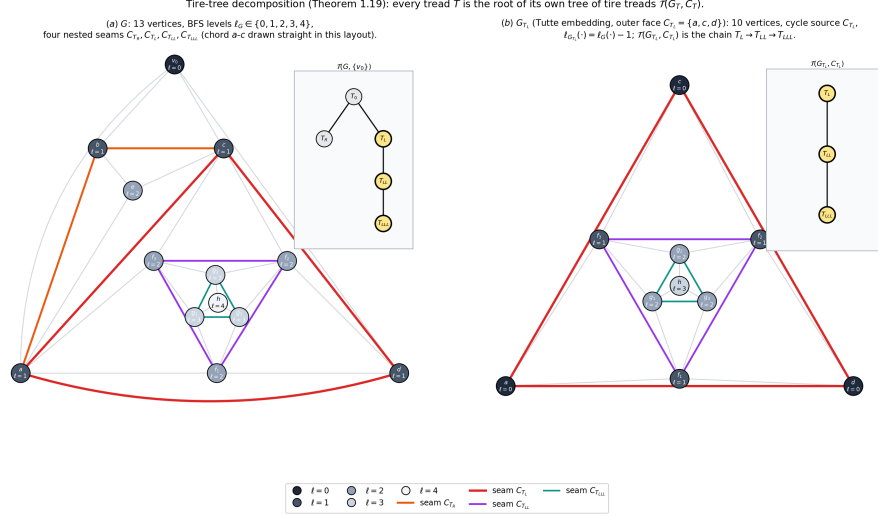


FIGURE 6. Tire-tree decomposition (Theorem 1.19) on a 13-vertex maximal planar example G with five BFS levels. (a) G with vertex source v_0 and $\ell_G \in \{0, 1, 2, 3, 4\}$; four nested seams are highlighted, $C_{T_R} = \{a, b, c\}$ (orange), $C_{T_L} = \{a, c, d\}$ (red, including the chord a - c shared with C_{T_R}), $C_{T_{LL}} = \{f_1, f_2, f_3\}$ (purple), $C_{T_{LLL}} = \{g_1, g_2, g_3\}$ (teal). Inset: the rooted tree of tire treads $\mathcal{T}(G, \{v_0\})$ branches at T_0 into the leaf T_R (containing e) and a chain $T_L \rightarrow T_{LL} \rightarrow T_{LLL}$ (the highlighted sub-tree). (b) The disk G_{T_L} inside the seam C_{T_L} , drawn standalone with C_{T_L} as cycle source and vertex labels rotated to match the new (cycle-source) role of the boundary triangle. $\ell_{G_{T_L}}(\cdot) = \ell_G(\cdot) - 1$ on $V(G_{T_L})$ (verified by the generator script), and $\mathcal{T}(G_{T_L}, C_{T_L})$ is the chain $T_L \rightarrow T_{LL} \rightarrow T_{LLL}$, iso to the highlighted sub-tree of (a).

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