

HEAWOOD RESTRICTIONS ON NESTED TIRE GRAPH DUALS

ERIC BAUERFELD

ABSTRACT.

1. INTRODUCTION

A classical theorem of Tait recasts the Four Colour Theorem in dual, edge-colouring terms: a plane triangulation G is properly 4-vertex-colourable if and only if its dual cubic graph G' is properly 3-edge-colourable. Thus a minimal counterexample to the Four Colour Theorem – a smallest triangulation admitting no proper 4-colouring – corresponds to a smallest cubic plane graph admitting no proper 3-edge-colouring.

This paper continues the series studying that structure through the lens of *nested level duals*. The foundational vocabulary — level sources, levels, the inner planar dual G' and its dual depth, and tire graphs — is developed in the companion paper [3]; we refer to that paper for those definitions and rely on them throughout. In particular we use, without restating, the notions of:

- *level source* S and G -vertex levels $\ell_G(v)$;
- the inner planar dual G' ([3, Definition 1.3]);
- *dual depth* $\delta_G(d_f)$ ([3, Definition 1.4]);
- *tire graph* $T = (B_{\text{out}}, O, E_{\text{ann}})$ with outer/inner boundaries and annular edges ([3, Definition 1.5]);
- the *tire-component lemma* ([3, Lemma 1.8]); and
- the *tire-tread partition theorem* ([3, Theorem 1.9]).

Throughout, $G = (V, E)$ is a plane maximal planar graph (a triangulation) with a fixed planar embedding Π_G . We write $|V| = n$, so $|E| = 3n - 6$ and G has $2n - 4$ triangular faces.

The classical input is Heawood's face-sum identity [1]: for any proper 3-edge-colouring of a cubic plane graph H , assigning each face of H a number in $\{+1, -1\}$ can be done so that the labels around every vertex of H sum to 0 (mod 3). In the triangulation G dual to H this becomes a $\{+1, -1\}$ labelling of the *faces* of G whose incident-face sum at every vertex of G vanishes mod 3. Our aim is to record what this restriction forces along the boundary cycles of a nested tire graph, and to formulate a chain-pigeonhole programme in this Heawood labelling parallel to the medial programme of [4].

2010 *Mathematics Subject Classification*. Primary .

Key words and phrases. plane graph, triangulation, plane depth, level edge, dual graph, tire graph, Heawood number.

2. CONNECTED TIRE CLUSTERS

The tire treads at a fixed depth partition the depth- d faces of G [3], but distinct depth- d tires need not be vertex-disjoint: a single vertex of G may lie on the source-side boundary of several depth- d tires at once (this occurs exactly when the depth- d faces around that vertex are split into more than one arc by depth- $(d-1)$ faces). We organise the depth- d tires by this sharing.

Lemma 2.1 (Same-depth tires meet only in vertices). *Let $T \neq T'$ be two distinct tire treads at the same depth d in $\mathcal{T}(G, S)$, arising from connected components C', C'' of the depth- d dual subgraph G'_d . Then T and T' share no edge of G ; any intersection $V(T) \cap V(T')$ consists of isolated vertices.*

Proof. An edge e of G shared by two depth- d annular faces f_1, f_2 is, by definition of the inner dual [3, Definition 1.3], a dual edge of G' joining d_{f_1} and d_{f_2} ; since $\delta(d_{f_1}) = \delta(d_{f_2}) = d$, this edge lies in G'_d , so d_{f_1} and d_{f_2} belong to the same component of G'_d . Hence no edge of G is shared by annular faces of two *different* components, and distinct depth- d tires share no edge. Their intersection is therefore a set of isolated vertices. \square

Definition 2.2 (Connected tire cluster). Fix a nested tire decomposition $\mathcal{T}(G, S)$ and a depth d . On the set of depth- d tire treads define the relation

$$T \sim T' \iff V(T) \cap V(T') \neq \emptyset.$$

A *connected tire cluster* at depth d is the subgraph of G

$$K = \bigcup_i T_i \subseteq G$$

obtained as the union (of underlying plane graphs) of the tires in a single connected component $\{T_i\}$ of the transitive closure of \sim . A cluster consisting of a single tire is *trivial*; the connected tire clusters at depth d partition the depth- d tires.

Remark 2.3. By Lemma 2.1 the constituent tires of a connected tire cluster are joined only at shared vertices, each of which is a cut vertex of K ; a connected tire cluster is thus a “cactus of tires” and is in general *not* itself a tire graph, since the annulus structure of [3, Definition 1.5] fails at each such pinch. The shared (cut) vertices are precisely the vertices that belong to more than one depth- d tire.

A single vertex may belong to several tires at one depth — the high-degree case where its depth- d faces split into many arcs — so the number of *tires* through a vertex is unbounded. Clustering collapses exactly this multiplicity: all tires through a vertex at a fixed depth share that vertex, hence lie in one cluster. The cluster count is therefore controlled.

Proposition 2.4 (A vertex meets at most two clusters). *Every vertex $v \in V(G)$ belongs to at most two connected tire clusters, namely at most one at each of the two consecutive depths $\ell_G(v) - 1$ and $\ell_G(v)$. In particular a source vertex ($\ell_G(v) = 0$) belongs to a single cluster.*

Proof. Write $\ell = \ell_G(v)$.

Step 1: every bounded face incident to v has dual depth $\ell - 1$ or ℓ . Let f be a bounded triangular face with $v \in V(f)$. Then $\delta_G(d_f) = \min_{u \in V(f)} \ell_G(u) \leq \ell_G(v) = \ell$. The other two vertices of f are adjacent to v in G , and the level function

$\ell_G(\cdot) = \text{dist}_G(\cdot, S)$ is 1-Lipschitz along edges, so each has level at least $\ell - 1$; hence $\delta_G(d_f) \geq \ell - 1$. Thus $\delta_G(d_f) \in \{\ell - 1, \ell\}$ (only $\delta_G(d_f) = 0$ when $\ell = 0$), so v bounds faces of, and therefore belongs to tires of, no depth other than $\ell - 1$ or ℓ .

Step 2: at each depth, all tires through v lie in one cluster. Fix $d \in \{\ell - 1, \ell\}$ and let T, T' be depth- d tires with $v \in V(T) \cap V(T')$. Then $V(T) \cap V(T') \neq \emptyset$, so $T \sim T'$ in the sense of Definition 2.2, and all depth- d tires containing v lie in a single connected component of \sim — one connected tire cluster K_d .

Combining the two steps, v belongs to at most the clusters $K_{\ell-1}$ and K_ℓ , i.e. to at most two connected tire clusters; when $\ell = 0$ only K_0 occurs. \square

3. HEAWOOD RESTRICTIONS ON THE TIRE DUAL

We work inside a fixed nested tire decomposition $\mathcal{T}(G, S)$ of G from a single-vertex level source S [3], and use the tire data $T = (B_{\text{out}}, O, E_{\text{ann}})$ with annular faces F_{ann} , outer boundary B_{out} , and inner boundary B_{in} ([3, Definition 1.5]). Since O is outerplanar, every vertex of a tire lies on B_{out} or on the inner-boundary walk B_{in} ; a tire has no interior vertices.

Definition 3.1 (Heawood face-labelling of a tire). A *Heawood face-labelling* of a tire graph T is a map

$$\lambda : F_{\text{ann}} \longrightarrow \{+1, -1\}$$

assigning a sign to each annular face of T . For a vertex $v \in V(T)$, write $F_{\text{ann}}(v) \subseteq F_{\text{ann}}$ for the set of annular faces of T incident to v , and define the *induced vertex value*

$$\lambda^*(v) := \sum_{f \in F_{\text{ann}}(v)} \lambda(f) \pmod{3} \in \{0, 1, -1\}.$$

The value $\lambda^*(v)$ is the *partial* face-sum at v taken over the annular faces of T alone, not over all faces of G incident to v .

Remark 3.2. Because a tire has no interior vertices, every annular face of T is incident to $B_{\text{out}} \cup B_{\text{in}}$, and a Heawood face-labelling is subject to *no* internal constraint: all $2^{|F_{\text{ann}}|}$ sign assignments are admissible. The Heawood restriction is felt only on the two boundary cycles, through the induced vertex values λ^* .

Definition 3.3 (Induced boundary sequences). Let λ be a Heawood face-labelling of T . Reading the vertices of B_{out} in clockwise order v_0, v_1, \dots, v_{p-1} , the *outer Heawood sequence* of (T, λ) is

$$\sigma_{\text{out}}(T, \lambda) := (\lambda^*(v_0), \dots, \lambda^*(v_{p-1})) \in \{0, 1, -1\}^p.$$

Reading the inner-boundary walk B_{in} in clockwise order w_0, \dots, w_{q-1} gives the *inner Heawood sequence* $\sigma_{\text{in}}(T, \lambda) \in \{0, 1, -1\}^q$. The *Heawood restriction relation* of T is the set

$$R_T := \{ (\sigma_{\text{out}}(T, \lambda), \sigma_{\text{in}}(T, \lambda)) : \lambda : F_{\text{ann}} \rightarrow \{+1, -1\} \}$$

of all (outer, inner) sequence pairs realisable by a single face-labelling, read up to rotation and the global sign-flip $\lambda \mapsto -\lambda$ (equivalently $\sigma \mapsto -\sigma$).

Definition 3.4 (Heawood compatibility across an interface). Let T be a tire and $T' \in \mathcal{T}(G, S)$ a child of T , so the outer boundary cycle $B_{\text{out}}^{(T')}$ coincides with a bounded face of $O^{(T)}$; let γ be this shared cycle, of length L , and let v range over

its vertices. Heawood face-labellings λ of T and λ' of T' are *compatible along γ* if at every shared vertex v ,

$$\lambda^*(v) + (\lambda')^*(v) \equiv 0 \pmod{3},$$

i.e. 0 is paired with 0 and +1 with -1 . Equivalently, the inner Heawood sequence of T on γ is the pointwise negation mod 3 of the outer Heawood sequence of T' on γ , after reversing one of the two clockwise readings to account for the opposite rotational senses in which T and T' traverse γ .

Remark 3.5. Call v *interior* if it is not incident to the outer face of Π_G . For an interior vertex every incident face is bounded, and compatibility along γ at v is exactly the statement that the incident-face sum at v — over the parent's annular faces together with the child's — vanishes mod 3:

$$(3.1) \quad \sum_{f \ni v} \lambda(f) \equiv 0 \pmod{3} \quad \text{for every interior vertex } v \in V(G),$$

the sum ranging over the bounded faces incident to v . The interfaces of $\mathcal{T}(G, S)$ are interior level cycles, so cluster compatibility only ever constrains interior vertices and is untouched by the outer face.

To pass from (3.1) to a colouring one must account for the outer face: an outer-boundary vertex is incident to the unbounded face f_∞ , whose label is omitted from the bounded sum. Extend λ by a single label $\lambda(f_\infty) \in \{+1, -1\}$ on f_∞ . Then a family of Heawood face-labellings that is pairwise compatible along every interface of $\mathcal{T}(G, S)$ assembles into a $\{+1, -1\}$ labelling of *all* faces of G for which $\sum_{f \ni v} \lambda(f) \equiv 0 \pmod{3}$ holds at every vertex — the outer-boundary vertices now carrying $\lambda(f_\infty)$ in their sum. This is Heawood's face-sum identity [1] for a proper 3-edge-colouring of the full cubic dual of G , hence (by Tait) a proper 4-vertex-colouring of G .

Why the programme runs between nested clusters. The vanishing condition (3.1) at a vertex v is a constraint on the *full* face-star of v . To run a pigeonhole between two objects — a child and a parent — we need that full sum to split as exactly two one-sided contributions, so that each vertex label is the combination of a single child value and a single parent value. This is true at the level of connected tire clusters, and *false* at the level of individual tires. Extend a Heawood face-labelling to a connected tire cluster K by labelling every annular face of every tire of K , and for $v \in V(K)$ write

$$\lambda_K^*(v) := \sum_f \lambda(f) \pmod{3},$$

the sum over the annular faces of K incident to v .

Proposition 3.6 (Two-sided cluster decomposition at a vertex). *Let $v \in V(G)$ have level $\ell = \ell_G(v)$, and let K_ℓ and $K_{\ell-1}$ be the at most two connected tire clusters containing v , of depths ℓ and $\ell-1$ respectively (Proposition 2.4). Then the bounded faces of G incident to v partition into the annular faces of K_ℓ at v and the annular faces of $K_{\ell-1}$ at v , and*

$$\sum_{f \ni v} \lambda(f) \equiv \lambda_{K_\ell}^*(v) + \lambda_{K_{\ell-1}}^*(v) \pmod{3}.$$

Each one-sided value $\lambda_{K_d}^*(v)$ is the complete sum over all depth- d faces at v , so the Heawood condition (3.1) at v reads

$$\lambda_{K_\ell}^*(v) + \lambda_{K_{\ell-1}}^*(v) \equiv 0 \pmod{3},$$

a pairing between the single child cluster K_ℓ and the single parent cluster $K_{\ell-1}$. (When $\ell = 0$, or when v bounds no depth- ℓ face, only one term is present.)

Proof. By Proposition 2.4 (Step 1) every bounded face incident to v has depth $\ell - 1$ or ℓ , partitioning the incident faces by depth; by Step 2 all depth- ℓ faces at v lie in the single cluster K_ℓ and all depth- $(\ell - 1)$ faces at v in $K_{\ell-1}$. Hence the depth- ℓ part is exactly the annular faces of K_ℓ at v , the depth- $(\ell - 1)$ part those of $K_{\ell-1}$, and summing λ over the two parts gives the identity; (3.1) is its vanishing. \square

Remark 3.7 (Failure at the tire level). Proposition 3.6 is what makes the binary parent/child pairing possible, and it requires the cluster. A vertex v may lie on many depth- ℓ tires — the unbounded case of Section 2 — and the per-tire value $\lambda^*(v)$ of Definition 3.1 then records only the faces of *one* tire at v , a fragment of v 's face-star. No single child tire carries the complete depth- ℓ sum, so the label $\sum_{f \ni v} \lambda(f)$ cannot be written as one child value plus one parent value, and per-tire compatibility (Definition 3.4) fails to assemble to (3.1). Clustering repairs this: Proposition 2.4 guarantees exactly one cluster meets v on each side, so $\lambda_{K_\ell}^*(v)$ is the complete child contribution and $\lambda_{K_{\ell-1}}^*(v)$ the complete parent contribution. Every vertex label is then realised as the combination of a single child-cluster value with a single parent-cluster value, and the pigeonhole programme below chains *nested connected tire clusters* rather than individual tires.

We write R_K for the *cluster Heawood restriction relation*: the set of (outer, inner) boundary Heawood sequence pairs realisable by a face-labelling of K , defined as in Definition 3.3 but with the outer and inner boundaries of the cluster and the complete one-sided values λ_K^* in place of a single tire's, read up to rotation and global sign-flip. By Proposition 3.6 two nested clusters are compatible along their shared interface exactly when the inner sequence of the parent is the pointwise negation mod 3 of the outer sequence of the child (after the orientation reversal of Definition 3.4).

Conjecture 3.8 (Heawood chain-pigeonhole principle). *There is a function $N(k)$ such that the following holds. Let*

$$K_0 \supset K_1 \supset \cdots \supset K_{N(k)}$$

be a nested chain of connected tire clusters in $\mathcal{T}(G, S)$ whose shared interfaces have length at most k . Then two adjacent cluster restriction relations $R_{K_i}, R_{K_{i+1}}$ in the chain admit compatible face-labellings along their shared interface, after rotation and global sign-flip. Equivalently, the chain contains a local gluing step that cannot be obstructed by disjoint Heawood boundary restrictions.

Conjecture 3.9 (Heawood cluster route to the Four Colour Theorem). *For every plane triangulation G and every level source S , the cluster Heawood restriction relations $\{R_K : K \text{ a connected tire cluster}\}$ admit a selection of face-labellings that is compatible along every cluster interface. By Proposition 3.6 and Remark 3.5 this yields a $\{+1, -1\}$ face-labelling of G satisfying (3.1), hence G is properly 4-vertex-colourable.*

4. THE CONSTRAINT FLOOR

A nested substructure constrains its outer interface through the set of Heawood boundary sequences it can realise. By the self-similarity of the tire decomposition ([3]), the region G_T enclosed by a tire's outer cycle, away from the source, is itself a triangulated disk; we ask how tightly any such disk can constrain its boundary. The achievable set below depends only on the disk triangulation, not on a tire-tree labelling.

Definition 4.1 (Achievable boundary set of a disk). Let D be a triangulated disk whose boundary is a simple n -cycle $C = (v_0, \dots, v_{n-1})$. Call a Heawood face-labelling $\lambda : F(D) \rightarrow \{+1, -1\}$ *interior-valid* if $\sum_{f \ni w} \lambda(f) \equiv 0 \pmod{3}$ at every interior vertex w of D (no condition on C). The *achievable boundary set* of D is

$$\Phi(D) := \{ (\lambda^*(v_0), \dots, \lambda^*(v_{n-1})) : \lambda \text{ interior-valid} \} \subseteq \{0, 1, -1\}^n.$$

Proposition 4.2 (Interior-free disks attain 2^{n-2}). *If D has no interior vertices then $|\Phi(D)| = 2^{n-2}$.*

Proof. A triangulation of the n -gon has an *ear*: a face (v_{i-1}, v_i, v_{i+1}) whose middle vertex v_i has face-degree 1, so $\lambda^*(v_i)$ equals that face's label and is read directly off the boundary sequence. Deleting the ear leaves a triangulation of the $(n-1)$ -gon inducing the restricted boundary sequence; inducting, the $n-2$ face labels are recovered injectively, so $\lambda \mapsto \lambda^*|_C$ is a bijection onto a set of size 2^{n-2} . \square

Lemma 4.3 (Un-stacking). *If v is a degree-3 interior vertex of D , deleting it and restoring its link triangle as a single face yields a disk D' with one fewer interior vertex and $\Phi(D') = \Phi(D)$.*

Proof. The constraint at v forces its three faces to a common value s , contributing $2s \equiv -s$ to each of the three link vertices; the restored triangle, labelled $-s$, reproduces that contribution at each, and $s \mapsto -s$ is a bijection on $\{+1, -1\}$. The resulting map is a bijection between interior-valid labellings of D and of D' preserving every boundary value, hence $\Phi(D) = \Phi(D')$. \square

Conjecture 4.4 (Constraint floor). *For every triangulated disk D with boundary an n -cycle, $|\Phi(D)| \geq 2^{n-2}$. Equivalently, no nested structure constrains the outer cycle below 2^{n-2} achievable Heawood sequences.*

Remark 4.5 (Status of Conjecture 4.4). Iterating Lemma 4.3 reduces any disk, Φ -faithfully and at fixed n , to one with no degree-3 interior vertex: either interior-free, where Proposition 4.2 gives exactly 2^{n-2} , or *irreducible* (every interior vertex of degree ≥ 4). Thus Conjecture 4.4 holds for the entire stacked (Apollonian) class and reduces to the irreducible case. Empirically it holds without exception over more than 10^4 disks, and *strictly*: every irreducible disk satisfies $|\Phi(D)| \geq \frac{5}{4} \cdot 2^{n-2}$, with equality at a single minimal-degree interior vertex; the wheel W_n gives $|\Phi(W_n)| = \lfloor 2^n/3 \rfloor$ and is *not* the minimiser. A counting balance makes the floor plausible — a disk with k interior vertices has $2k + n - 2$ faces (Euler) but only k interior constraints, so the linear free dimension $k + n - 2$ grows with depth — but this is only heuristic: $|\Phi(D)|$ is *not* monotone in k (inserting a degree-4 vertex can shrink it), it merely never drops below 2^{n-2} . Two natural elementary proofs, a Φ -non-increasing vertex reduction and a direct $(n-2)$ -face transversal, both provably fail; a proof appears to need a global argument on the Boolean / mod-3 structure of Φ .

Remark 4.6. Two consequences. First, $\Phi(D)$ is a $\mathbb{Z}/3$ zonotope — a projected cube, sign-closed but not a $\text{GF}(3)$ subspace — and at the interior-free value it has size 2^{n-2} with affine hull of dimension $n - 2$. Second, granting Conjecture 4.4, the floor is exponential in the interface length n , so a maximally-constraining child still offers 2^{n-2} outer options, and the gluing of Conjecture 3.8 has the least slack at *short* interfaces (e.g. $n = 4$ leaves 4 options) and is easy at long ones; the difficulty of the programme is concentrated at short level cycles.

REFERENCES

- [1] P. J. Heawood, *On the four-colour map theorem*, Quart. J. Pure Appl. Math. **29** (1898), 270–285.
- [2] E. Bauerfeld, *Plane Depth*, manuscript (math-research repository), 2026.
- [3] E. Bauerfeld, *Nested Tire Decompositions of Plane Triangulations*, manuscript (math-research repository), 2026.
- [4] E. Bauerfeld, *Medial Tire Decompositions of Plane Triangulations*, manuscript (math-research repository), 2026.
- [5] E. Bauerfeld, *Coloring Nested Tire Dual Graphs*, manuscript (math-research repository), 2026.