LEVEL SWITCHING

ERIC BAUERFELD

Abstract.

1. Introduction

2. Definitions

Throughout, $G = (V, E)$ is a plane maximal planar graph (a triangulation) with a fixed planar embedding Π_G. We write $|V| = n$, so $|E| = 3n - 6$ and G has $2n - 4$ triangular faces.

Definition 2.1 (Level source)**.** A *level source* of G is either:

- a face F of G (all vertices of F are level-0 sources), or
- a vertex v of degree 3 (the singleton $\{v\}$ is a level-0 source).

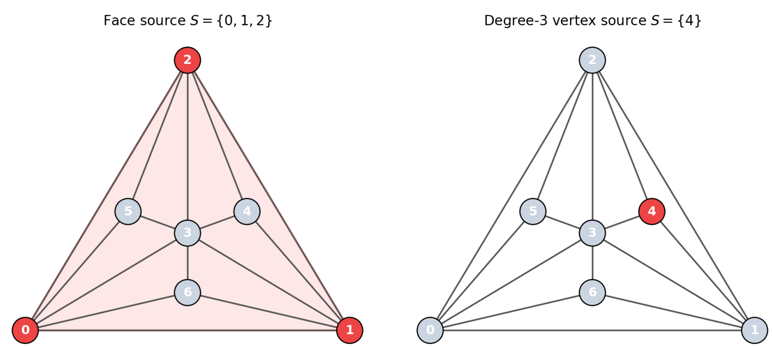

Figure 1. The two kinds of level source on a 7-vertex triangulation T (K_4 with vertices $4, 5, 6$ stacked into the three interior faces). Left: the face source $S = \{0, 1, 2\}$ (level-0 vertices are the corners of the highlighted triangle). Right: the degree-3 vertex source $S = \{4\}$.

Definition 2.2 (Levels)**.** Given a level source $S \subseteq V$, the *level* of $v \in V$ is $\ell_G(v) = \mathrm{dist}_G(v, S)$, the graph distance from v to the nearest source vertex.

Definition 2.3 (Level cycle)**.** A *level cycle* of G (with respect to a level source S) is a simple cycle in G all of whose vertices have the same level.

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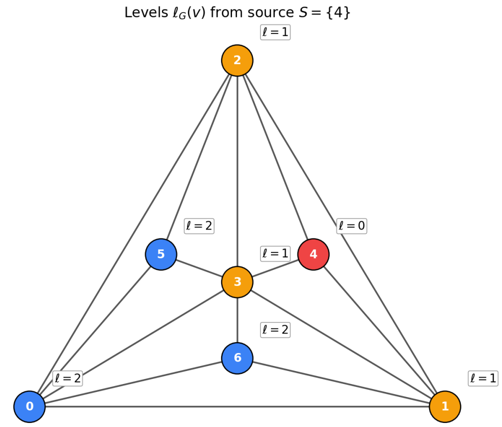

FIGURE 2. BFS levels from the degree-3 vertex source $S = \{4\}$. The source is level 0, its three neighbours are level 1, and the remaining vertices are level 2. Colour encodes the level.

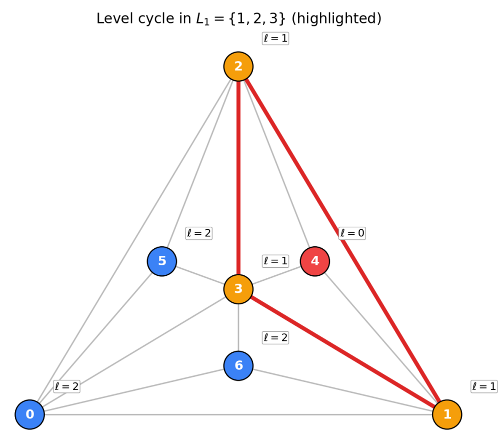

FIGURE 3. A level cycle in the triangulation of Figure 2. The triangle 1−2−3 is a simple cycle whose three vertices all lie at level 1, so it is a level cycle at level 1.

Definition 2.4 (Edge switch)**.** Let G be a triangulation with level source S, and let $e = uv$ be an edge of a level cycle of G. The *edge switch* at e is the edge flip on e: writing uvw and uvx for the two triangular faces of G containing e, the edge uv is removed and the edge wx is added. As with any edge flip, the result is a triangulation on the same vertex set provided w and x are non-adjacent in G.

Definition 2.5 (Parity subgraph)**.** Let G be a triangulation with level source S, and let G' be a triangulation on the same vertex set as G. The *even parity subgraph* $E_{G,S}(G')$ is the subgraph of G' induced by $\{v \in V : \ell_G(v) \equiv 0 \pmod 2\}$. The *odd parity subgraph* is defined analogously for odd ℓ_G.

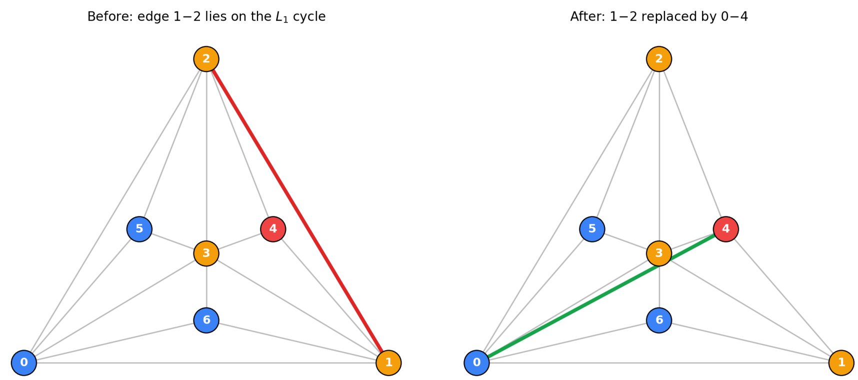

FIGURE 4. An edge switch on the level cycle of Figure 3. The chosen cycle edge $1-2$ is shared by the triangular faces $(0,1,2)$ and $(1,2,4)$; the switch deletes $1-2$ (red, left) and inserts $0-4$ (green, right). Vertex colours indicate the original levels in G.

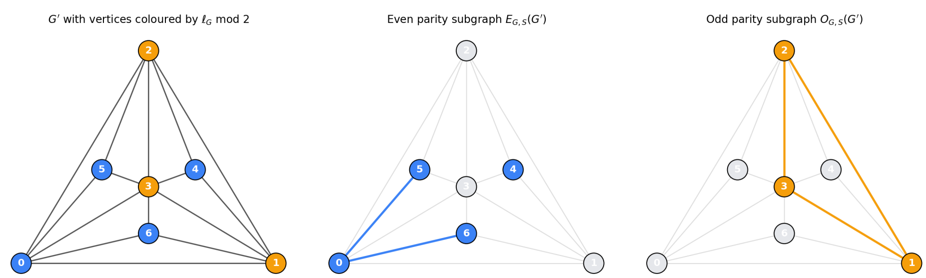

FIGURE 5. Parity subgraphs of $G' = T$ with respect to the level structure of Figure 2 (here we take $G = G' = T$). Left: T with vertices coloured by ℓ_G mod 2 (blue = even, orange = odd). Middle: the even parity subgraph $E_{G,S}(G')$, induced on $\{0,4,5,6\}$; only edges with both endpoints even appear. Right: the odd parity subgraph $O_{G,S}(G')$, induced on $\{1,2,3\}$; the highlighted triangle shows that $O_{G,S}(G')$ is not bipartite for this choice of G'.

Definition 2.6 (Facial depth)**.** Let L_k be drawn with the outerplanar embedding inherited from Π_G, let D be the dual graph of this drawing with the outer face removed, and let $\mathcal{B}$ be the set of inner faces of L_k whose bounding level cycle contains at least one edge of the outer cycle of L_k. The *facial depth* of an inner face F of L_k is

$$\operatorname{depth}(F) \;=\; \min_{F' \in \mathcal{B}} \operatorname{dist}_D(F, F'),$$

with the convention $\operatorname{depth}(F) = \infty$ if no such F' exists. An inner face is *isolated* if $\operatorname{depth}(F) \geq 1$.

Definition 2.7 (Surface switch)**.** A *surface switch* is an edge switch (Definition 2.4) applied to an edge incident to two level cycles, one of facial depth d and the other of facial depth $d-1$.

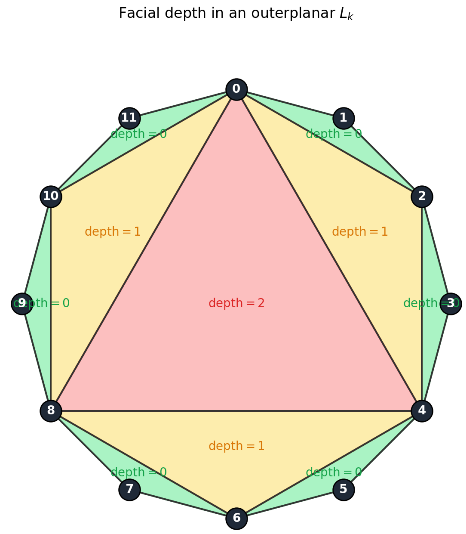

FIGURE 6. Facial depths in a maximal outerplanar graph on 12 vertices. The six green ear-triangles share an edge with the outer 12-cycle and so lie in $\mathcal{B}$ (depth 0). The three yellow "in-between" triangles $(0, 2, 4), (4, 6, 8), (0, 8, 10)$ have only diagonal edges but each is dual-adjacent to ears, giving them depth $= 1$. The central triangle $(0, 4, 8)$ is also all-diagonal; its dual neighbours are the three depth-1 triangles, so it is isolated with depth $= 2$.

Definition 2.8 (Balanced surface switch)**.** Let σ be a surface switch on an edge $e = uv$ separating an inner face F of L_k of depth $d \geq 1$ from an adjacent inner face $F' = uvx$ of depth $d-1$. We say σ is *balanced* if each of the two edges of $\partial F'$ other than uv (namely ux and vx) either lies on the outer cycle of L_k or is shared with an inner face of L_k of depth $d-2$.

When $d = 1$ the condition reduces to "both ux and vx lie on the outer cycle of L_k", because no inner face has depth -1; in that case F' is a triangular "ear" hanging off uv.

3. OUTERPLANARITY OF LEVEL COMPONENTS

For each integer $k \geq 0$ and each (G, S), write L_k for the subgraph of G induced by the level-k vertices. A *level component* of G (with respect to S) is a connected component of some L_k.

Theorem 3.1. *For every plane triangulation G and every level source S of G, every level component of G is outerplanar.*

Proof. Since every subgraph of an outerplanar graph is outerplanar, it suffices to show that each level subgraph L_k is outerplanar. For $k = 0$, L_0 is either a single vertex (when S is a degree-3 vertex) or the triangle bounding the source face (when S is a face), both outerplanar.

Fix $k \geq 1$ and let D_k be the drawing of L_k inherited from Π_G. Let F^* be the face of D_k containing the source. Suppose for contradiction that some $u \in L_k$ does not lie on ∂F^*, so u lies on the boundary of some other face of D_k. Take any path P in G from $v_0 \in S$ to u. As a curve in Π_G, P starts in F^* and ends at a point off ∂F^*, so it must transition from F^* to a different face of D_k; in a planar embedding this can happen only at a vertex of D_k, that is, at a level-k vertex w on P. Either $w \neq u$ (so P has length $\geq \mathrm{dist}_G(S, w) + 1 \geq k + 1$), or $w = u$ (contradicting $u \notin \partial F^*$). Since every S-to-u path has length $\geq k + 1$, $\mathrm{dist}_G(S, u) \geq k + 1$, contradicting $u \in L_k$. □

Lemma 3.2. *Let C be a level component of G with respect to S, drawn with the outerplanar embedding inherited from Π_G, and let D be its inner-face dual. If F is an inner face of C with* $\mathrm{depth}(F) = d > 0$*, then F is dual-adjacent to an inner face F' with* $\mathrm{depth}(F') = d - 1$.

Proof. By Theorem 3.1, C is outerplanar, so the inner-face dual D is a forest (a standard fact; a tree when C is 2-connected).

Each leaf F_ℓ of D contains a single interior edge of C, so the remaining edges of ∂F_ℓ lie on the outer cycle of C. In particular F_ℓ has at least one outer-cycle edge, so $F_\ell \in \mathcal{B}$ and $\mathrm{depth}(F_\ell) = 0$. Hence every tree component of D contains an element of $\mathcal{B}$, so the depths of all of its vertices are finite.

Choose a shortest path $F = F_0, F_1, \ldots, F_d = F^*$ in D from F to some $F^* \in \mathcal{B}$ realising $\mathrm{depth}(F) = d$. The suffix $F_1, \ldots, F_d$ witnesses $\mathrm{depth}(F_1) \leq d - 1$. If $\mathrm{depth}(F_1) \leq d - 2$, prepending the edge $F F_1$ to a witnessing path would give $\mathrm{depth}(F) \leq d - 1$, contradicting $\mathrm{depth}(F) = d$. Hence $\mathrm{depth}(F_1) = d - 1$, and we may take $F' := F_1$. □

Proposition 3.3. *Let σ be a balanced surface switch on the edge $e = uv$ separating F (depth $d \geq 1$) from $F' = uvx$ (depth $d - 1$), and let $G' = G - uv + wx$ be the result of the underlying edge flip, with uvw, uvx the two triangular faces of G at uv. Then in L'_k (the level-k subgraph of G', with the level assignment of G):*

(1) the level cycle ∂F is destroyed; and
(2) one or two new inner faces appear in L'_k, each of depth exactly $d - 1$.

Proof. The flip removes uv from G, so ∂F is no longer a cycle of L'_k, proving (1). For (2) we split on whether the new edge wx re-enters L_k.

Case (i): $\{w, x\} \not\subseteq L_k$. Then $L'_k = L_k - uv$. The faces F and F' merge into a single new inner face $\widetilde{F}$ with boundary $(\partial F \cup \partial F') \setminus \{uv\}$. The dual neighbours of $\widetilde{F}$ in L'_k are exactly the former neighbours of F and F' other than each other; in particular they include all inner faces previously adjacent to F' across ux or vx, whose depths are at most $d - 2$ by Lemma 3.2 applied to F' (when $d \geq 2$). Thus $\mathrm{depth}(\widetilde{F}) \leq d - 1$.

For the matching lower bound, every neighbour of $\widetilde{F}$ has depth $\geq d - 2$ (neighbours inherited from F have depth $\geq d - 1$; neighbours inherited from F' have depth $\geq d - 2$). When $d \geq 2$, neither F nor F' has an outer-cycle edge, so neither does $\widetilde{F}$, giving $\mathrm{depth}(\widetilde{F}) \geq d - 1$. When $d = 1$, $F' \in \mathcal{B}$ and its outer-cycle edge (necessarily distinct from the interior edge uv) survives on $\partial\widetilde{F}$, so $\widetilde{F} \in \mathcal{B}'$ and $\mathrm{depth}(\widetilde{F}) = 0 = d - 1$. In either case $\mathrm{depth}(\widetilde{F}) = d - 1$, giving the unique new face required by (2).

Case (ii): $\{w, x\} \subseteq L_k$**.** Then $F = uvw$ and $F' = uvx$ are triangular faces of L_k, and $L'_k = L_k - uv + wx$. The chord wx splits the quadrilateral $\partial(F \cup F')$ into two triangular faces $A = uwx$ and $B = vwx$ of L'_k. We show $\mathrm{depth}(A) = \mathrm{depth}(B) = d - 1$.

By symmetry it suffices to handle A. The dual neighbours of A in L'_k are A_{uw} (the inner face across uw, unchanged from L_k), A_{ux} (the inner face across ux, unchanged), and B (across the new edge wx). By balancedness of σ applied to the edge ux:

- if ux lies on the outer cycle of L_k, it remains on the outer cycle of L'_k, so $A \in \mathcal{B}'$ and $\mathrm{depth}(A) = 0$ (which equals $d - 1$ because the balanced-with-outer-cycle case forces $d = 1$); or
- if A_{ux} is an inner face, balancedness gives $\mathrm{depth}(A_{ux}) = d-2$, so $\mathrm{depth}(A) \leq 1 + (d - 2) = d - 1$.

For the lower bound in the second sub-case ($d \geq 2$): A's edges are uw (an edge of F, interior because F has depth $d \geq 1$), wx (new, not on the outer cycle), and ux (interior in this sub-case), so $A \notin \mathcal{B}'$. Moreover every neighbour of A has depth $\geq d - 2$: A_{uw} inherits depth $\geq d - 1$ from being a former neighbour of F, A_{ux} has depth $d - 2$, and B has depth $\geq d - 2$ by the same argument applied symmetrically. Therefore $\mathrm{depth}(A) \geq d - 1$, and combined with the upper bound, $\mathrm{depth}(A) = d - 1$. ☐

When does a balanced surface switch exist? For a chord uv of a maximal outerplanar graph, the *span* of uv is the minimum, over the two arcs from u to v on the outer cycle, of the number of outer-cycle vertices strictly between them.

Observation 3.4. For $d = 1$, an inner face F admits a balanced surface switch on some edge iff at least one edge of F has span 1 in the outer cycle of L_k. The opposite triangle across that edge – using the single outer-cycle vertex between its endpoints – is then an ear of F in $\mathcal{B}$, satisfying the $d = 1$ form of Definition 2.8.

The smallest maximal-outerplanar configuration violating this is a 9-vertex outer cycle triangulated so that the unique interior face $F = (0, 3, 6)$ has spans $(2, 2, 2)$ on its three edges (Figure 7). Each depth-0 neighbour of F carries exactly one outer-cycle edge, not two, so none qualifies as an ear of F; no balanced surface switch is available.

Preprocessing toward balanced switches. When F has depth $d \geq 1$ but admits no balanced surface switch, perform a single (unbalanced) surface switch on any edge of F shared with a depth-$(d - 1)$ neighbour. By Proposition 3.3 the result is at least one new depth-$(d - 1)$ face; in Case (ii) it is accompanied by a new depth-d face A that replaces F as the next candidate. The hope is that the resulting A admits a balanced surface switch, or that iterating the preprocessing eventually exposes one.

Example 3.5. On the 9-vertex example, the (unbalanced) surface switch on edge $uv = 03$ – with $F' = (0, 2, 3)$, third vertex $x = 2$, and $w = 6$ – flips $03 \mapsto 26$ in G and produces $A = (0, 2, 6)$ at depth 1. The new face has spans $(1, 3, 2)$ on its edges, and the ear $(0, 1, 2)$ across the span-1 edge 02 is now a balanced surface-switch target on A (Figure 8).

We do not have a general termination theorem. The natural candidate monovariant for $d = 1$ is the minimum span among edges of the current depth-1 face that

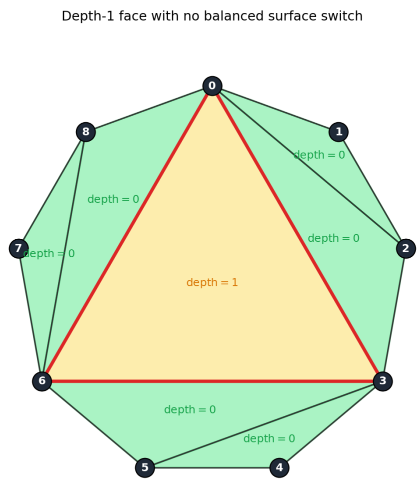

FIGURE 7. 9-vertex maximal outerplanar L_k. $F = (0, 3, 6)$ has depth $= 1$ and all three of its edges have span 2, so none of F's depth-0 neighbours is an ear. No balanced surface switch is available on F.

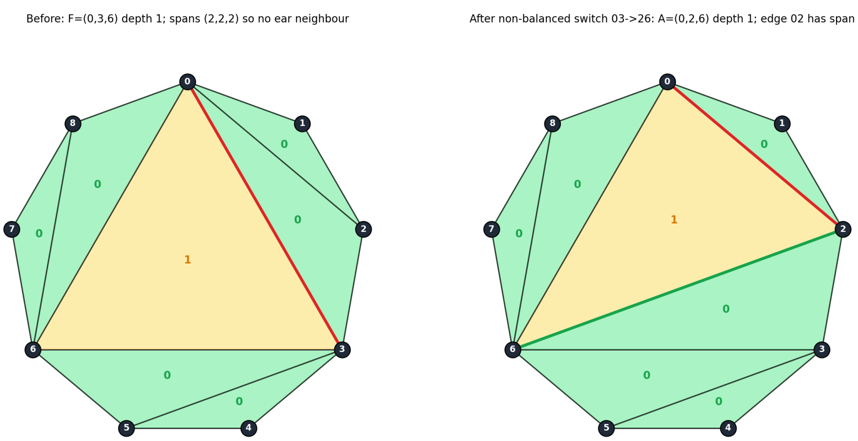

FIGURE 8. One step of preprocessing on the 9-vertex example. Left: $F = (0, 3, 6)$ has no edge of span 1; the chosen surface-switch edge $uv = 03$ (red) is unbalanced. Right: after the switch $03 \mapsto 26$ (green), the new depth-1 face $A = (0, 2, 6)$ has its edge 02 (red) at span 1, exposing the ear $(0, 1, 2)$ as a balanced surface-switch target.

are shared with a depth-0 neighbour; in Example 3.5 this drops from 2 to 1 in a single step. Whether such a monovariant strictly decreases under every unbalanced surface switch – and a corresponding statement for $d \geq 2$, where balancedness depends on depth-$(d-2)$ structure rather than just spans – remains open.

The $d \geq 2$ analog and recursive lopsidedness. For $d \geq 2$ the obstruction to a balanced surface switch is no longer "F has no edge of span 1": it is recursive. We say a depth-$(d-1)$ neighbour $F' = uvx$ of F is *lopsided* if exactly one of its non-F neighbours has depth $d-2$ (the other being deeper or an interior face of depth $d-1$). F admits a balanced surface switch iff at least one depth-$(d-1)$ neighbour is not lopsided.

The analog of the 9-vertex example at $d = 2$ is a 21-vertex configuration where the unique depth-2 face $F = (0, 7, 14)$ has three depth-1 neighbours $(0, 3, 7)$, $(7, 10, 14)$, $(14, 17, 0)$, each lopsided: their depth-1 "deep side" is a degree-3 face $(3, 5, 7)$, $(10, 12, 14)$, $(17, 19, 0)$ that itself reaches depth 0 via two ears. So the obstruction at F is one layer of lopsidedness; after a single preprocessing step the new depth-2 face $(3, 7, 14)$ sees the previously-hidden balanced descender as a direct neighbour and the algorithm terminates immediately.

Stacking lopsidedness yields a 24-vertex example (Figure 9) where every depth-1 neighbour of F is lopsided *and* the depth-1 degree-3 face inside each arm (G_i) is itself lopsided. Two preprocessing steps are needed before a balanced switch becomes available: the active depth-2 face migrates from $(0, 8, 16)$ to $(2, 8, 16)$ to $(4, 8, 16)$, at which point the *innermost* depth-1 face $(4, 6, 8)$ – whose two non-F neighbours $(4, 5, 6)$ and $(6, 7, 8)$ are both ears – becomes a direct neighbour and the balanced condition is satisfied. After the balanced switch, 10 further balanced switches drive every face to depth 0.

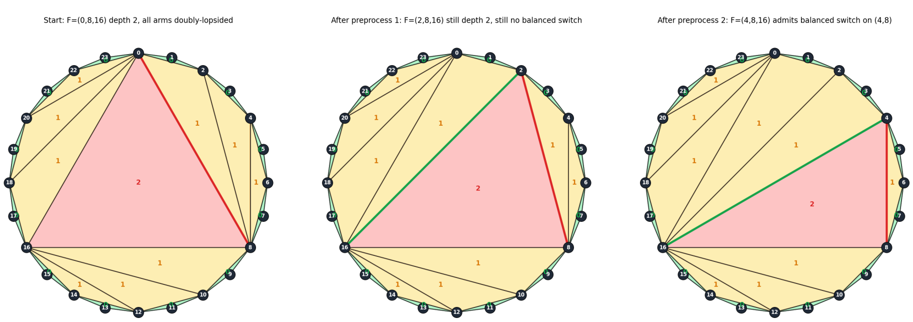

Figure 9. Recursive lopsidedness at $d = 2$. Left: $F = (0, 8, 16)$ depth 2, every arm doubly-lopsided. Middle: one preprocessing switch $(0, 8) \mapsto (2, 16)$ exposes the first lopsided layer; the new depth-2 face $(2, 8, 16)$ still has no balanced switch. Right: a second preprocessing switch $(8, 2) \mapsto (4, 16)$ reaches the inner balanced face $K_0 = (4, 6, 8)$, whose two non-F neighbours are both ears; the depth-2 face $(4, 8, 16)$ now admits a balanced surface switch on edge $(4, 8)$.

Empirical termination. On every tested configuration, iterated preprocessing terminates and the algorithm

while max-depth face F has depth$(F) > 0$: do a balanced switch if available, else preprocess drives every face to depth 0. The observed step count is

configuration	n	$d_{\max}$	total switches
no-balanced $d = 1$ (Figure 7)	9	1	4
singly-lopsided $d = 2$ (Figure 9 left only)	21	2	8
doubly-lopsided $d = 2$ (Figure 9)	24	2	13

Each preprocessing step appears to advance the active maximum-depth face one vertex along the lopsided arm of the chosen depth-$(d-1)$ neighbour, peeling off one layer of recursive lopsidedness. The remaining open question is to identify the monovariant that captures this: a candidate is the total number of triples (F, F', F'') where $F' \in N(F)$ is lopsided and $F'' \in N(F')$ is its depth-$d-1$ "deep side". We do not yet have a proof that this strictly decreases under every unbalanced surface switch on a maximum-depth face.

What the natural monovariants do not give us. The most obvious candidate – the lexicographic depth signature $\left(\#\{F : \text{depth}(F) \geq k\}\right)_{k\geq 1}$ – is *weakly* but not strictly decreasing: a balanced surface switch removes the level cycle bounding F and creates one or two cycles of depth $d-1$, so each balanced switch strictly decreases the signature in some component. But an unbalanced surface switch in Case (ii) removes one depth-d face and creates one depth-$(d-1)$ face plus one depth-d face, so the signature is unchanged. The same holds for the simpler sum $\sum_F \text{depth}(F)$: on the 24-vertex example of Figure 9 the sum is 11 at every preprocessing step, dropping only when balanced switches begin.

A finer candidate is the dual-tree distance from the active maximum-depth face F to the nearest face $F^\bullet$ that admits a balanced surface switch as a depth-d face. Empirically, with the preprocessing edge chosen along the path from F to $F^\bullet$, this distance strictly decreases by 1 per preprocessing step; combined with the strict drop in the depth signature at each balanced step, (signature, tree-distance-to-$F^\bullet$) then becomes a lexicographically decreasing monovariant. We do not have a proof that $F^\bullet$ always exists, nor a recipe to identify it without look-ahead.

Empirical termination on random configurations. Beyond the constructed examples, we ran the iterated algorithm (balanced switch when available, otherwise preprocess via a deterministic edge choice) on random triangulations of polygons of size n up to 24 (10-20 trials per size). Every trial terminated, with the worst-case total step count growing roughly as $O(n^2)$: about 13 steps at $n = 24$, an order of magnitude more by $n = 40$. With a random edge choice the algorithm still terminates empirically but takes substantially more steps, suggesting that the deterministic strategy (advancing toward a known $F^\bullet$) matters for efficient termination.

Question 3.6. Does iterated preprocessing always reach a balanced surface switch in finitely many steps? More specifically: in every maximal outerplanar L_k with $d_{\max} \geq 1$, does there exist a face $F^\bullet$ that admits a balanced surface switch – and if so, can it always be reached from the current maximum-depth face by a preprocessing path of length bounded by the dual-tree diameter of L_k?

4. Reachability via edge switches

We now sketch a positive termination argument that bypasses the local question of balanced surface switches entirely: instead of insisting that each switch strictly improve facial depth, we show that any L_k can be transformed by edge switches into a configuration in which every face has depth 0. Throughout this section we adopt the *stable-labelling convention*: the level $\ell_G(v)$ of each vertex is fixed at the start of the procedure (by BFS from S in the initial triangulation G) and reused thereafter, even after edge switches modify the triangulation. In particular, the level-k subgraph L_k of the current triangulation always means "the subgraph induced on the vertices labelled k at the start".

Two cases on the layer below k. We split on whether any L_k-face has a higher-level vertex in its interior in the planar embedding inherited from Π_G.

Case 1: every inner face of L_k is a triangle and contains no vertex of level $\geq k+1$ in its interior.

Under this hypothesis, for every chord $uv \in L_k$ the two G-triangles at uv have their third vertices in L_k (since the interior of the two L_k-faces adjacent to uv in Π_G contains no other vertex of G). The edge switch at uv is therefore always in Case (ii) of Proposition 3.3, and acts as a flip of the chord uv in L_k regarded purely as a maximal outerplanar graph.

Maximal outerplanar graphs on n labelled vertices (arranged on a common outer cycle) are exactly triangulations of a convex n-gon. The set of such triangulations, with chord flips as edges, is the 1-skeleton of the associahedron and is connected; in fact any two triangulations are joined by $O(n)$ chord flips [1]. A *fan triangulation* – the triangulation obtained by adding chords from a single apex vertex v_0 to every other vertex – has every inner triangle bounded by an outer-cycle edge (namely the side opposite v_0 in that triangle), so every face of a fan triangulation lies in $\mathcal{B}$ and has depth 0.

Combining: in Case 1, L_k can be transformed into a fan triangulation by $O(n)$ edge switches, after which every face has depth 0.

Case 2: some L_k-face F has a vertex of level $\geq k+1$ in its interior.

Pick any edge uv of ∂F. The G-triangle at uv on the F-side has its third vertex w inside F, so w is a vertex of level $\geq k+1$ and in particular $w \notin L_k$. The edge switch at uv is therefore in Case (i) of Proposition 3.3: the edge uv is removed from L_k, no new edge is added to L_k, and the face F merges with the L_k-face on the opposite side of uv into a single larger face. The number of inner faces of L_k strictly decreases by 1.

Combining.

Theorem 4.1. *Under the stable-labelling convention, every L_k can be transformed by edge switches into a configuration in which every inner face of L_k has facial depth* 0*, in $O(n)$ edge switches.*

Proof sketch. Apply Case 2 repeatedly while L_k has any inner face with a higher-level vertex in its interior. Each application reduces the number of inner faces of L_k by 1, so after at most $|L_k| - 2 \leq n - 2$ such steps we reach one of:

- A configuration satisfying the hypothesis of Case 1, in which case Case 1 finishes the job in $O(n)$ flips by reaching a fan triangulation.
- A configuration in which L_k has only one inner face – i.e., L_k consists of only its outer cycle, with no chords. The unique inner face is bounded by all n outer-cycle edges, so it lies in $\mathcal{B}$ and has depth 0.

Both outcomes leave every face at depth 0. The total step count is at most $(n - 2) + O(n) = O(n)$. $\square$

Remark 4.2. Theorem 4.1 settles the existence question Question 3.6 affirmatively in the following sense: *some* sequence of edge switches drives every face to depth 0 in $O(n)$ steps. It does not, however, identify the sequence by a local rule (the leaf-distance algorithm of Section 4's preceding discussion), and in particular the question of which *rule* produces such a sequence without backtracking remains open.

References

[1] D. D. Sleator, R. E. Tarjan, W. P. Thurston. *Rotation distance, triangulations, and hyperbolic geometry.* Journal of the American Mathematical Society, 1988.