A double-contraction reductio at a degree-5 vertex via Kempe chains as Heawood face-chains

Status: exploratory strategy note. Records a proof skeleton and the single lemma it reduces to; nothing here is proved. Companion to `boundary_restriction_structure.tex` *and* `paper.tex`*.*

Setup: the move

Let G be a minimal counterexample to the Four Colour Theorem (a smallest triangulation with no proper 4-vertex-colouring). By Euler G has a vertex v of degree ≤ 5; degrees ≤ 4 are classically reducible, so take $\deg(v) = 5$. The link of v is a pentagon a, x, b, y, z (consecutive neighbours adjacent). For a non-adjacent (diagonal) pair, say a, b, the *double contraction* contracts both edges (a, v) and (b, v), identifying $a = v = b$ into a single vertex. There are 5 diagonal pairs.

The double contraction is a proper minor with two fewer vertices, hence a smaller planar graph G'; by minimality G' is 4-colourable. It stays a simple triangulation iff each contracted edge is non-separating (its endpoints have exactly two common neighbours).

Kempe chains as Heawood face-chains

Work in the cubic dual G' (vertices = faces of G, so the Heawood ± 1 labels sit on dual vertices). Tait-colour the edges with the three nonzero Klein-4 elements $\{a, b, c\}$ (edge colour = colour-difference across the primal edge). Then:

Observation (Kempe chain = Tait cycle = flip-set)**.** *A primal Kempe chain (component of two vertex-colour classes of G) corresponds to a connected component of the two-edge-colour subgraph $\{a, b\}$ of the dual. That subgraph is 2-regular, so it is a disjoint union of cycles; each such cycle alternates $a, b, a, b, \ldots$ and is therefore* ***always even****. The cycle is literally a cyclic chain of faces of G. A Kempe swap reverses the rotational order (a, b, c) at exactly the vertices on the cycle, so the set of faces whose Heawood sign* flips *under the swap is precisely the Kempe chain.*

Observation (Sign alternation = same-side rule)**.** *Even-ness does* not *force the Heawood signs to alternate along the cycle. At each cycle vertex the third (off-cycle, colour-c) edge points either inside or outside the region bounded by the cycle. For consecutive vertices A, B:*

$$\text{same side (both in / both out)} \Rightarrow \text{signs flip}; \qquad \text{opposite side} \Rightarrow \text{signs repeat.}$$

(Local chirality computation: the four in/out cases give $+, -$ / $-, +$ / $+, +$ / $-, -$.) Hence the Heawood sign sequence around a Tait cycle is determined, up to one global sign, by the in/out pattern of the third edges; it is fully alternating iff there are zero side-switches, i.e. all third edges lie on one side.

The transport hypothesis (L1)

Conjecture (Chain transport, claimed for all triangulations)**.** *The double contraction preserves the relevant (crossing) Kempe/Heawood chain structure as it propagates up the nested tire decomposition: the two chains anchored at the pentagon survive, consistently, through every tire interface above v.*

The reductio (two nested contradictions)

We are *not* claiming G' is uncolourable, and we are not claiming intertwined Kempe chains are impossible (they occur, e.g. in the Errera graph). The reducible object is *forced* intertwining.

Inner. Suppose intertwining is the *only* way to colour G' (every colouring is intertwined at v). By transport the two crossing chains pass through every interface above; if uncrossing is excluded at each interface, the tire restriction relations R_{K} collapse to their "crossed-only" sub-relations. Contract each interface **along the chain** to obtain a strictly smaller triangulation H''. If the crossed-only sub-relations admit no compatible gluing on H'', then H'' is uncolourable and smaller than G — contradicting minimality.

Outer. Therefore G' admits a non-intertwined colouring; it uncrosses at v, frees a colour for v, and lifts to a colouring of G — contradicting that G is a counterexample. $\square$ (modulo the lemma below)

What it all reduces to

Claim (the only open content)**.** *Under the crossed-only collapse, the surviving Heawood boundary sequences at a (forced-short) tire interface have empty pointwise-negation gluing on H'' — not merely small, genuinely empty.*

This is where the 2^{n-2} constraint floor and the mod-3 side-pattern must do real work: the floor guarantees that *smallness alone* never empties a relation, so the emptiness must come from the chains pinning the sub-relation down, not from short interface length.

Open points / to pin next

- **Define H'' precisely.** Current candidate: contract each tire interface along the transported chain (the global analogue of the local double contraction). Confirm this is a definite, strictly smaller triangulation.
- **Lift is not automatic.** "Non-intertwined" must be defined as "admits a swap collapsing the pentagon a, x, b, y, z to ≤ 3 colours," or freeing v fails even after uncrossing (with $c(a) = c(b)$ the pentagon can still show 4 colours).
- **Errera oracle.** Run the whole construction against the Errera graph (and Fritsch / Kittell). These are colourable but Kempe-intertwined; the hypothesis "every colouring intertwined" must *fail* for them, so the construction must decline. *Why* it declines is exactly the mechanism the Claim must deny in the counterexample case.
- **Prove or break L1** as a statement about all triangulations.