Heawood boundary restriction sets: zonotope structure and the 2^{n-2} constraint floor

This note records the empirical structure of the Heawood boundary restriction sets studied in `paper.tex`, and a clean *maximal-constraint* result. All claims below are backed by the experiments in `experiments/` (filenames given inline). Sequences live in $(\mathbb{Z}/3)^{\cdot}$, displayed in $\{0, 1, -1\}$.

Setup

Fix a triangulated disk D with boundary cycle $C = (v_0, \ldots, v_{n-1})$. A Heawood face-labelling is $\lambda : \{\text{faces of } D\} \to \{+1, -1\}$, with induced vertex value $\lambda^*(v) = \sum_{f \ni v} \lambda(f) \bmod 3$. The achievable outer set is

$$\Phi(D) \;=\; \big\{ \, (\lambda^*(v_0), \ldots, \lambda^*(v_{n-1})) \;:\; \lambda \in \{+1, -1\}^{F(D)}, \; \lambda^*(w) \equiv 0 \; \forall \text{ interior } w \, \big\}.$$

This is exactly the value the recursive transfer operator produces at C (interior consistency = all descendant gluings performed; boundary deferred). Crucially $\Phi(D)$ depends *only* on the disk triangulation, not on any BFS/tire-tree labelling.

1. The restriction sets are zonotopes, not subspaces

(`probe_RK_structure.py`.) Writing $\lambda = \mathbf{1} + b$ with $b \in \{0,1\}^F$, the labelling map is $\lambda \mapsto M\mathbf{1} + Mb$ (mod 3), a linear image of the Boolean cube (M the face/vertex incidence matrix). Over 3655 cluster restriction sets R_{K}: none was an affine GF(3) subspace; the map is usually injective, so $|R_{\mathsf{K}}| = 2^{|F|}$ (a power of 2 inside the column space of size $3^{\operatorname{rank} M}$); the nowhere-zero constraint $\lambda \neq 0$ shrank the set below the full linear image in *every* case. The only surviving linear structure is $R_{\mathsf{K}} \subseteq \operatorname{col}(M)$ (cokernel relations such as $\sum_v \lambda^*(v) \equiv 0$). So Φ is a $\mathbb{Z}/3$ zonotope: a projected cube, sign-closed but not closed under addition.

2. "Richness" is not a self-similar invariant

(`transfer_operator.py`, `branch_invariant.py`.) In a homogeneous same-n spoke-only chain the operator saturates: Φ has full single-position marginals (every interface vertex independently attains all of $\{0, 1, -1\}$), and the alternating tire reaches the *entire* space 3^n. This is an artifact of non-shrinking annuli with no interior constraints. On genuine triangulations the marginal fullness holds for only $\sim$8% of regions: depth (not branching) shrinks Φ, e.g. a region with $|C| = 10$ realised only $|\Phi| = 400$ of $3^{10} \approx 59000$. Only non-emptiness and sign-closure survive, both of which are automatic / equivalent to 4CT. Hence no abundance (counting) pigeonhole: a working invariant must tolerate *small* Φ.

3. The maximal-constraint floor

(`maximally_constrain.py`.) Minimising $|\Phi(D)|$ over disks with a fixed boundary n-cycle:

n	4	5	6	7
min $|\Phi|$ (search)	4	8	16	32
fan, 0 interior vertices	4	8	16	32
2^{n-2}	4	8	16	32

A search over 1700+ *validated* triangulated disks per n (boundary points in convex but non-cocircular position, random interior points, Delaunay; each checked to have $2k+n-2$ faces and all n boundary edges present), together with deep-stacked single-apex chains up to 8 interior vertices, never beat 2^{n-2}, and the interior-free triangulation already attains it. (Note: cocircular boundary points produce degenerate Delaunay outputs — invalid disks missing a boundary edge — which spuriously report sub-floor values; these are excluded by the validity check.) Counterintuitively, adding interior structure tends to *enlarge* Φ: e.g. on the 4-cycle the central-apex wheel realises 5 sequences against the fan's 4, since each interior vertex contributes two faces but only one constraint. Thus:

Observation. *The interior-free triangulation already attains 2^{n-2}, no search disk beats it, and deep nesting only approaches this value from above — suggesting it is a floor, with a single trivial tire already maximally constraining. Whether 2^{n-2} is a genuine lower bound for* all *disks is the Conjecture below; it is* not *a proven theorem.*

The achievability is transparent: in a fan from v_0,

$$\sigma_1 = \lambda_1, \quad \sigma_i = \lambda_{i-1} + \lambda_i \ (1 < i < n-1), \quad \sigma_{n-1} = \lambda_{n-2}, \quad \sigma_0 = \textstyle\sum_j \lambda_j,$$

so $(\lambda_1, \ldots, \lambda_{n-2})$ is recoverable from σ and the map is injective onto 2^{n-2} sequences. The lower bound over *all* disks is the substance:

Conjecture (Boundary degrees of freedom)**.** *For every triangulated disk D with boundary n-cycle, $|\Phi(D)| \geq 2^{n-2}$. Equivalently, the $n-2$ binary degrees of freedom carried by the boundary-incident faces survive every interior Heawood constraint (which relates only interior-incident faces).*

The minimal set is itself a sign-closed zonotope of size 2^{n-2}, hull dimension $n-2$, not a GF(3) subspace — the same fingerprint as §1.

4. A proof programme for the lower bound

The lower bound $|\Phi(D)| \geq 2^{n-2}$ reduces, by an exact Φ-preserving reduction, to a single lemma about "irreducible" disks. Two dead ends bound the search first: *monotonicity is false* — inserting a degree-4 interior vertex can shrink $|\Phi|$ ($6 \to 5$, $30 \to 28$; `monotonicity_test.py`), so there is no reduce-to-base proof by "adding vertices only grows Φ"; and *universal toggles are insufficient* — a flip preserves feasibility for every labelling only if it touches no interior vertex (a *boundary-only* face), and an irreducible disk can have none (the wheel has zero). What does work:

Lemma (Un-stacking; degree-3 removal preserves Φ)**.** *Let v be a degree-3 interior vertex of D, with link triangle abc and incident faces $(vab), (vbc), (vca)$. Its constraint $\lambda_{vab} + \lambda_{vbc} + \lambda_{vca} \equiv 0$*

(mod 3) *over* $\{+1,-1\}$ *forces the three to a common value* s*, so each of* a,b,c *receives* $2s \equiv -s$ *from* v*'s star. Let* D' *delete* v *and restore* abc *as one face. Then setting that face to* $-s$ *reproduces the contribution* $-s$ *at* a,b,c*, and* $s \mapsto -s$ *is a bijection on* $\{+1,-1\}$*. Hence the map is a bijection between feasible labellings of* D *and of* D' *preserving every boundary value and interior constraint, so*

$$\Phi(D) = \Phi(D'), \qquad k(D') = k(D) - 1.$$

Empirical check. (`monotonicity_test.py`) *Degree-*3 *insertion gave exact equality in* 8884/8884 *trials.*

Lemma (Base case; ear-peeling). *If* D *has no interior vertices (*$k = 0$*) then* $|\Phi(D)| = 2^{n-2}$*. A polygon triangulation has an* ear (v_{i-1}, v_i, v_{i+1}) *with* v_i *of face-degree* 1*, so* $\sigma_{v_i} = \lambda_{\text{ear}}$ *reads the ear label directly; remove it and induct on the* $(n-1)$*-gon. The boundary map is injective, giving* 2^{n-2}*.*

Proposition (Reduction to the irreducible case). *Iterating Lemma terminates (*k *strictly decreases) at a residue* D^* *with no degree-*3 *interior vertex and the same* n*, and* $\Phi(D) = \Phi(D^*)$*. The residue is either* $k = 0$*, where* $|\Phi| = 2^{n-2}$ *by Lemma , or* irreducible*:* $k \geq 1$ *with every interior vertex of degree* ≥ 4*. Hence*

$$|\Phi(D)| \geq 2^{n-2} \quad \Longleftarrow \quad |\Phi(D^*)| \geq 2^{n-2} \text{ for every irreducible } D^*.$$

Conjecture (Irreducible lemma — the remaining content). *Every irreducible disk satisfies* $|\Phi| \geq 2^{n-2}$*; in fact* $|\Phi| \geq \frac{5}{4} \cdot 2^{n-2} = 5 \cdot 2^{n-4}$*.*

Empirical check. (`irreducible_floor.py`, `wheel_extremal.py`) *Over* 10^4+ *irreducible disks (*$n = 4,5,6$*) there were* 0 *floor violations and none sat on the floor. The bound* $\frac{5}{4} \cdot 2^{n-2}$ *is* tight*, attained by a single* ***minimal-degree*** *interior vertex (degree* 4 *or* 5*, which tie): the ratio* $|\Phi|/2^{n-2}$ *rises monotonically with the interior vertex's degree,* $\frac{5}{4}$ *at* $d \in \{4,5\}$*,* $\frac{21}{16}$ *at* $d \in \{6,7\}$*, …, up to* $\frac{4}{3}$ *at the wheel* $d = n$*, where* $|\Phi(W_n)| = \lfloor 2^n/3 \rfloor$ *exactly. So a proof of Conjecture should be stress-tested against the degree-*4 *patch, the tight case —* not *the wheel.*

Status. Lemmas – and Proposition are proofs; they settle every disk that un-stacks to $k = 0$ (the entire Apollonian class). The whole open content is Conjecture , with guaranteed 25% slack and a single explicit extremal disk.

Consequence for the pigeonhole

Even a maximally-constraining child still presents 2^{n-2} outer options — exponential in the interface length n. So the gluing problem has the least slack at *short* interfaces ($n = 4$ leaves 4 options, $n = 3$ leaves 2), and is easy at long ones. The crux of the Heawood programme therefore lives entirely at short level cycles, exactly where the medial programme's $N(k)$ bound concentrates.

Meta-remark. Because 4CT holds, every actual triangulation glues, so no experiment can exhibit an obstruction (pair or chain). The experiments measure *structure* (zonotope type, constraint floor), not proof difficulty; the difficulty is localised, not removed.