HUMANS SUFFICE: A NOVEL PROOF OF THE FOUR COLOR THEOREM

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Dedicated to those who doubt the machine

Abstract.

1. Kempe's Proof (Valid Portion)

1.1. **Setup.** Kempe's strategy, published in 1879, follows a *minimal counterexample* argument. Suppose, for contradiction, that there exists a planar graph requiring 5 colors. Among all such graphs, let G be one with the fewest vertices. Then every planar graph with fewer vertices than G is 4-colorable, but G itself is not.

1.2. **Every Planar Graph Has a Vertex of Degree at Most 5.**

Lemma 1.1. *Every planar graph has at least one vertex of degree* ≤ 5.

Proof. Let G be a connected planar graph with V vertices, E edges, and F faces. By Euler's formula,

$$V - E + F = 2.$$

Since every face is bounded by at least 3 edges and each edge borders at most 2 faces, we have $2E \geq 3F$, hence $F \leq \frac{2E}{3}$. Substituting into Euler's formula:

$$V - E + \frac{2E}{3} \geq 2 \implies E \leq 3V - 6.$$

If every vertex had degree ≥ 6, then $2E \geq 6V$, so $E \geq 3V$, contradicting $E \leq 3V - 6$. Therefore at least one vertex has degree ≤ 5. □

Since G is a minimal counterexample, it must contain a vertex v of degree at most 5. Kempe argued by cases on the degree of v.

1.3. **Cases of Degree at Most 3.** Suppose v has degree ≤ 3. Remove v from G to obtain the graph $G - v$. Since $G - v$ has fewer vertices than G, it is 4-colorable by minimality. Fix such a 4-coloring. Now reinsert v: its at most 3 neighbors occupy at most 3 of the 4 colors, so at least one color remains available for v. This yields a valid 4-coloring of G, a contradiction.

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1.4. **Case of Degree 4.** Suppose v has degree exactly 4 with neighbors a, b, c, d appearing in cyclic order around v in the planar embedding. Remove v and 4-color $G - v$ by minimality. If the four neighbors do not all receive distinct colors, then at least one color is unused among them, and we may assign that color to v, giving a contradiction.

So assume a, b, c, d receive all four distinct colors; call them 1, 2, 3, 4 respectively. Define a *Kempe chain* to be a maximal connected subgraph whose vertices are colored with exactly two specified colors.

Consider the Kempe chain K_{13} containing a (using colors 1 and 3).

- **Case 1**: c is not in K_{13}. Swap colors 1 and 3 throughout K_{13}. This is still a valid coloring of $G - v$, and now a receives color 3, so color 1 is free for v.
- **Case 2**: c is in K_{13}. Then there is a path of alternating colors 1 and 3 from a to c in the planar embedding. Because a and c alternate around v with b and d, this path separates b from d in the plane. Therefore b and d lie in different Kempe chains for colors 2 and 4. Swap colors 2 and 4 in the chain containing b; now b receives color 4, freeing color 2 for v.

In both cases we obtain a valid 4-coloring of G, a contradiction.

2. Resolution of Degree 5 Case

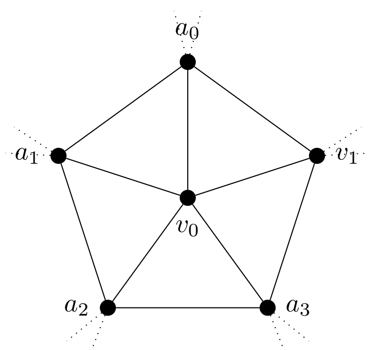

Figure 1. The neighborhood of a degree-5 vertex v_0: a 5-cycle on a_0, a_1, a_2, a_3, v_1 with v_0 adjacent to each.

Let G be a minimum example of a maximal planar graph requiring 5 colors, and let v_0 be a graph in G of degree 5, and let v_1 be a neighbor of v_0.

Lemma 2.1. *Let G^* be the subgraph of G obtained by deleting the edge $\{v_0, v_1\}$, and let G' be the graph obtained by merging vertices a_0 and a_3 in G^*. Then there is at least one proper 4-coloring of $G*$ which assigns v_0 and v_1 the same color and a_0 and a_3 the same color.*

Proof. Since G' has fewer vertices than G, it admits a proper 4-coloring ϕ by minimality. Any such ϕ must assign v_0 and v_1 the same color: if they received distinct colors, ϕ would induce a proper 4-coloring of G^* in which $v_0 \neq v_1$, and re-inserting edge $\{v_0, v_1\}$ would yield a proper 4-coloring of G, contradicting minimality. The induced coloring Φ of G^* assigns a_0 and a_3 the common color $\phi(a_{03})$ by construction, so Φ is the required coloring. $\square$

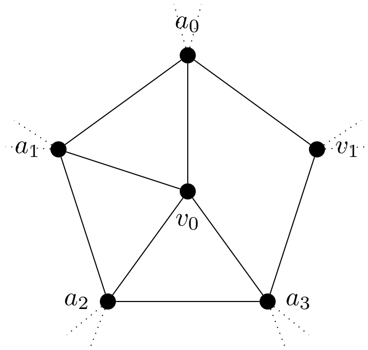

FIGURE 2. The neighborhood of v_0 and v_1 in G^*: G with the edge $\{v_0, v_1\}$ deleted, so the spoke to v_1 is absent.

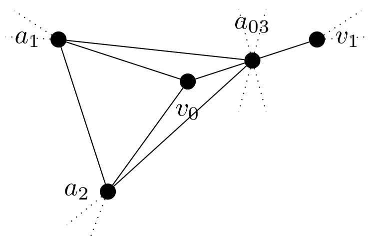

FIGURE 3. The neighborhood of v_0 and v_1 in G': G^* with a_0 and a_3 merged into a_{03}.

Lemma 2.2. *Let Φ be a proper 4-coloring of $G*$ induced by a proper 4-coloring of G', let $\alpha = \Phi(v_0) = \Phi(v_1)$, and let $\beta = \Phi(a_0) = \Phi(a_3)$. Then for each of the two remaining colors γ and δ, there exists a Kempe chain in G' connecting v_0 to v_1 using colors $\{\alpha, \gamma\}$ and $\{\alpha, \delta\}$ respectively.*

Proof. We show v_0 and v_1 lie in the same $\{\alpha, \gamma\}$-Kempe chain; the argument for $\{\alpha, \delta\}$ is identical.

By the previous lemma, Φ must exist. Suppose for contradiction that v_0 and v_1 lie in *different* $\{\alpha, \gamma\}$-Kempe chains. Swapping colors α and γ throughout v_0's chain yields a new proper 4-coloring Φ' of $G*$ in which $\Phi'(v_0) = \gamma$ while $\Phi'(v_1) = \alpha$. Since $\gamma \neq \alpha$, the coloring Φ' assigns v_0 and v_1 distinct colors, which allows us to properly color G. Therefore v_0 and v_1 lie in the same $\{\alpha, \gamma\}$-Kempe chain, and by the same argument they lie in the same $\{\alpha, \delta\}$-Kempe chain. □

Lemma 2.3. *Let Φ be a proper 4-coloring of $G*$ induced by a proper 4-coloring of G', let $\alpha = \Phi(v_0) = \Phi(v_1)$, let $\beta = \Phi(a_0) = \Phi(a_3)$, let $\delta = \Phi(a_2)$, and let $\gamma = \Phi(a_1)$. Then the coloring Φ' obtained by doing a Kempe switch on the $\{\beta, \delta\}$-Kempe chain incident to a_0 must assign $\alpha = \Phi'(v_0) = \Phi'(v_1)$, $\beta = \Phi'(a_3)$, $\delta = \Phi'(a_0) = \Phi'(a_2)$, and $\gamma = \Phi'(a_1)$ (IE the coloring for all vertices except a_0 remains the same)*

Proof. The kempe switch will assign $\Phi'(a_0) = \delta$ by definition. By the previous lemma, there must be a $\{\alpha, \gamma\}$ kempe chain incident to v_0, v_1, and a_1. Therefore the $\{\beta, \delta\}$ component incident to a_2 and a_3 cannot be connected to the $\{\beta, \delta\}$

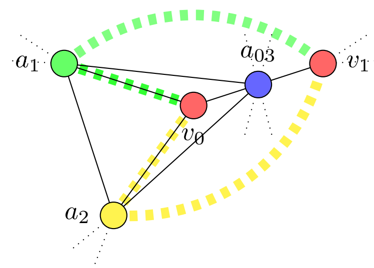

FIGURE 4. A proper 4-coloring of G' with $\phi(v_0) = \phi(v_1) = \text{red}$, $\phi(a_{03}) = \text{blue}$, $\phi(a_1) = \text{green}$, $\phi(a_2) = \text{yellow}$. By the previous lemma, v_0 and v_1 must lie in both the $\{\text{red}, \text{green}\}$-Kempe chain (dashed green, passing through a_1) and the $\{\text{red}, \text{yellow}\}$-Kempe chain (dashed yellow, passing through a_2).

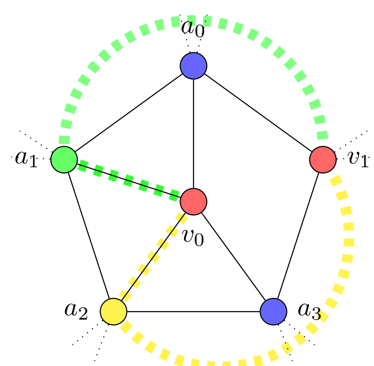

FIGURE 5. The same coloring lifted to G^*: $\phi(v_0) = \phi(v_1) = \text{red}$, $\phi(a_0) = \phi(a_3) = \text{blue}$, $\phi(a_1) = \text{green}$, $\phi(a_2) = \text{yellow}$. The $\{\text{red}, \text{green}\}$- and $\{\text{red}, \text{yellow}\}$-Kempe chains connecting v_0 to v_1 pass through a_1 and a_2 respectively.

component incident to a_0, so the kempe switch we do to obtain Φ' will not recolor vertices a_2 and a_3. Since the $\{\beta, \delta\}$ kempe switch also maintains vertices assigned α and γ in Φ, Φ' will not recolor vertices a_1, v_0, and v_1. Therefore $\alpha = \Phi'(v_0) = \Phi'(v_1)$, $\beta = \Phi'(a_3)$, $\delta = \Phi'(a_0) = \Phi'(a_2)$, and $\gamma = \Phi'(a_1)$. $\square$

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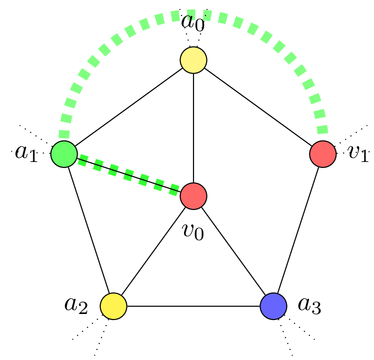
a_0
a_1
v_1
v_0
a_2
a_3