

# Edge 3-colorings of small outerplanar graphs with $\Delta \leq 3$ : a menagerie

## Setup

For a graph  $G$  write  $P_e(G, k)$  for the number of proper  $k$ -edge-colorings of  $G$  (= the chromatic polynomial of the line graph  $L(G)$  evaluated at  $k$ ). Throughout this note  $\Delta(G)$  denotes the maximum degree of  $G$ ; we are interested in  $k = 3$  and  $\Delta(G) \leq 3$ , with  $G$  outerplanar.

There is no *universal* closed form for  $P_e(G, 3)$  on the class of subcubic outerplanar graphs, but the class is small enough that every  $G$  in it decomposes along its block-cut tree into building blocks each of which admits a closed-form count. The building blocks form a short menagerie.

## The menagerie

### 1. Path $P_n$ ( $n$ vertices, $n - 1$ edges; $\Delta \leq 2$ )

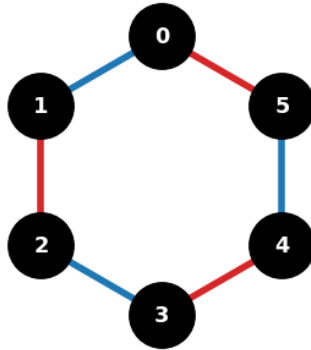


The line graph  $L(P_n)$  is the path  $P_{n-1}$ , so

$$P_e(P_n, k) = P_{\text{vert}}(P_{n-1}, k) = k(k-1)^{n-2}, \quad P_e(P_n, 3) = 3 \cdot 2^{n-2}.$$

(For a single edge,  $n = 2$ , the count is 3; for two edges in a row,  $n = 3$ , the count is 6.)

### 2. Cycle $C_n$ ( $\Delta = 2$ )

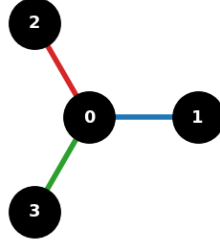


The line graph  $L(C_n) = C_n$ , so

$$P_e(C_n, k) = (k-1)^n + (-1)^n(k-1), \quad P_e(C_n, 3) = 2^n + 2(-1)^n.$$

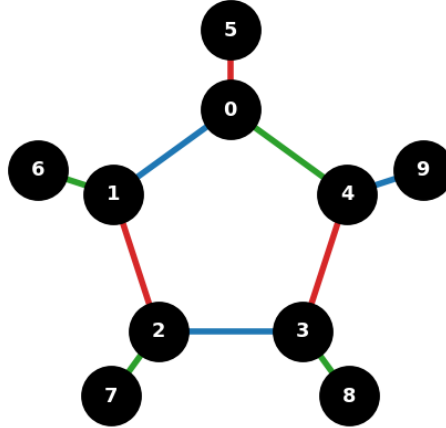
For even  $n$  the count is  $2^n + 2$ ; for odd  $n$  it is  $2^n - 2$ .

### 3. Star $K_{1,3}$ (a single $\Delta = 3$ vertex)



Three pairwise-incident edges at a single vertex must carry three distinct colors, so  $P_e(K_{1,3}, 3) = 3! = 6$ . More generally  $P_e(K_{1,d}, k) = k(k-1) \cdots (k-d+1)$ , which is positive iff  $k \geq d$ , i.e. iff  $k \geq \Delta(G)$ .

### 4. Corona $C_n \circ K_1$ (cycle with one leaf per cycle vertex; $\Delta = 3$ )



Each cycle vertex  $v$  has degree 3 in  $C_n \circ K_1$ : its two cycle edges must carry distinct colors and its leaf must carry the unique remaining third color. So the leaf coloring is *forced* by the cycle coloring, and

$$P_e(C_n \circ K_1, 3) = P_e(C_n, 3) = 2^n + 2(-1)^n.$$

This is the form of the partial tire dual  $D(T)$  in the spoke-only case (with  $L = n + m$ ).

### 5. Trees with $\Delta \leq 3$

A tree  $T$  on  $n$  vertices has  $|E(T)| = n - 1$  edges; its line graph  $L(T)$  is a *block graph* (every block is a clique). Edge-color a tree greedily by processing edges in BFS order from a leaf: when an

edge  $\{u, v\}$  is added, the only colors forbidden are those already used on the edges incident to its already-colored endpoint. Hence at any vertex of degree  $d$ , when the  $d$ -th edge is added there are exactly  $k - (d - 1)$  choices. For  $k = 3$  and  $\Delta \leq 3$ :

$$P_e(T, 3) = 3 \prod_{e \in E(T) \setminus \{e_0\}} (3 - d_e),$$

where  $e_0$  is the first edge processed and  $d_e$  is the number of already-processed edges incident to the new endpoint of  $e$  (between 1 and 2, since  $\Delta \leq 3$ ). In practice this gives a clean product depending only on the degree sequence of  $T$ .

## 6. Two-connected outerplanar with $\Delta \leq 3$ : cycle, possibly with a matching of chords

The 2-connected outerplanar graphs are polygons (with optional chords). Each chord raises the degree of its two endpoints by 1, and a polygon vertex already has degree 2 from the cycle. So for  $\Delta \leq 3$  the chords must form a *matching* of polygon vertices — no vertex can be an endpoint of two chords. In particular the simplest non-cycle case is a polygon with a *single* chord, denoted  $\theta(1, p, q)$ : two paths of lengths  $p$  and  $q$  between two trivalent vertices, plus a direct edge between those two trivalent vertices. Equivalently it is  $C_{p+q}$  with a chord joining the two cycle vertices at distance  $p$  apart on the polygon.

*[ $\theta(1, p, q)$ : polygon  $C_{p+q}$  with one chord; both chord endpoints have degree 3, other polygon vertices have degree 2.]*

$\theta(1, p, q)$  is outerplanar (the chord lies inside the polygon, and all polygon vertices are on the outer face) and subcubic.

**Closed form.** For each choice of chord color  $c \in \{1, 2, 3\}$ , the two cycle edges incident to each chord endpoint must lie in  $\{x, y\} = \{1, 2, 3\} \setminus \{c\}$ . Conditioning on whether the two pairs of cycle-edge colors at the chord endpoints agree or disagree, and using the transfer matrix  $T = J - I$  on  $K_3$  (eigenvalues 2 and  $-1$  with multiplicities 1 and 2), one finds

$$P_e(\theta(1, p, q), 3) = \frac{2^{p+q} - 2^p(-1)^q - 2^q(-1)^p + 10(-1)^{p+q}}{3}.$$

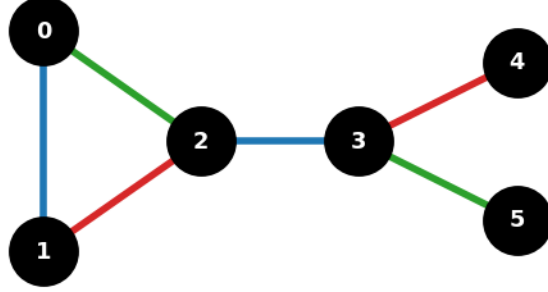
Sanity checks:

- $p = q = 2$ :  $\theta(1, 2, 2) = K_4 - e$ , formula gives  $(16 - 4 - 4 + 10)/3 = 6 = P_e(K_4 - e, 3)$ .
- $p = q = 3$ :  $\theta(1, 3, 3)$ , formula gives  $(64 + 8 + 8 + 10)/3 = 30$ .
- $p = q = 6$ :  $\theta(1, 6, 6)$  (= the interior dual subgraph of the partial tire dual for the barbell- $O$  tire of Figure 4 of the main paper), formula gives  $(4096 - 64 - 64 + 10)/3 = 1326$ .

The formula has been verified empirically against Sage's chromatic polynomial routine for all  $p, q \in \{2, 3, 4, 5, 6\}$ .

More generally, a polygon with  $r$  chords forming a matching has chromatic polynomial computable by the same transfer-matrix idea along the  $r+1$  paths between consecutive chord endpoints on the polygon, with a product constraint at each chord endpoint.

## 7. Block–cut decomposition



A general subcubic outerplanar graph is a union of cycle-blocks and edge-blocks glued at cut vertices. By case 6 each 2-connected block is a cycle; the remaining blocks are single edges (i.e. tree edges). At a cut vertex  $v$  of degree  $d_v \in \{2, 3\}$ , the colors of the  $d_v$  edges incident to  $v$  must be pairwise distinct. Counting  $P_e(G, 3)$  for the whole graph  $G$  amounts to counting colorings of each block independently and then enforcing the distinct-color constraint at every cut vertex. For  $k = 3$  and  $\Delta \leq 3$  this gives

$$P_e(G, 3) = \prod_{B \text{ cycle block of } G} P_e(B, 3) \cdot \prod_{B \text{ edge block of } G} P_e(B, 3) / \prod_{v \text{ cut vertex}} (\text{normalization at } v),$$

where the normalization corrects the over- or under-counting at the cut-vertex constraint. For each cycle-block  $B = C_n$  contributing  $2^n + 2(-1)^n$  proper 3-edge-colorings, and each edge-block contributing 3, this product is computable in time linear in  $|V(G)| + |E(G)|$ .