Fiber distributions for spoke-only tire face connectors: enumeration data for $n = 3, \dots, 12$

What this is

This note is the output of Step 1 of the action items in `birkhoff_heesch_reducibility.tex`: a brute-force enumeration of the fiber distribution $N(T'_{f'}; \sigma)$ for small spoke-only tire annular face connectors, framed as a $k = 3$ edge-coloring of the graph

$$G_n \;:=\; C_n \;+\; (\text{one pendant edge at each cycle vertex}),$$

which is the spoke-only $T'_{f'}$ when $n = m+k$ (the boundary cycle lengths of the underlying tire sum to n). Script: `experiments/tire_fiber_enumeration.py`; raw data: `experiments/fiber_data.json`.

Setup recap

A proper edge 3-coloring of G_n assigns colors to n cycle edges and n pendant edges so that the three edges incident to each cycle vertex v_i get distinct colors. Since $\deg(v_i) = 3$ and we have exactly 3 colors, the pendant color s_i is uniquely determined by the two cycle-edge colors at v_i as $s_i = \{1, 2, 3\} \setminus \{c(e_{i-1}), c(e_i)\}$. Therefore the map

$$\{\text{proper edge 3-colorings of } G_n\} \;\longrightarrow\; \Sigma := \{1, 2, 3\}^n, \qquad (c, s) \mapsto s,$$

is well-defined, and

$$P_e(G_n, 3) \;=\; |\{\text{cycle colorings}\}| \;=\; 2^n + 2(-1)^n \;=\; \sum_{\sigma \in \Sigma} N(G_n; \sigma).$$

The realisable support is $\mathcal{C} := \{\sigma : N(G_n; \sigma) > 0\}$. “A-reducibility” (every σ realisable) corresponds to $\mathcal{C} = \Sigma$.

Main table

n	P_e	$\lvert\mathcal{C}\rvert$	$\lvert\Sigma\rvert = 3^n$	$\lvert\mathcal{C}\rvert/\lvert\Sigma\rvert$	A-red.?	max N	min N	fiber size dist.
3	6	6	27	0.222	no	1	1	1×6
4	18	15	81	0.185	no	2	1	1×12, 2×3
5	30	30	243	0.123	no	1	1	1×30
6	66	63	729	0.086	no	2	1	1×60, 2×3
7	126	126	2187	0.058	no	1	1	1×126
8	258	255	6561	0.039	no	2	1	1×252, 2×3
9	510	510	19683	0.026	no	1	1	1×510
10	1026	1023	59049	0.017	no	2	1	1×1020, 2×3
11	2046	2046	177147	0.012	no	1	1	1×2046
12	4098	4095	531441	0.008	no	2	1	1×4092, 2×3

Two empirical findings

Observation (Fiber-size dichotomy). *For every $n \in \{3, \dots, 12\}$ examined, every fiber has size 1 or 2. The only σ with $N(\sigma) = 2$ are the three constant configurations $\sigma = (a, a, \dots, a)$ for $a \in \{1, 2, 3\}$, and only when n is even. For odd n, no constant σ is realisable.*

This is straightforward from the cycle structure: a constant σ forces every cycle edge to lie in $\{1, 2, 3\} \setminus \{a\}$, so the cycle alternates between two colors — possible only for even n, and in two ways. For non-constant σ, the cycle coloring is uniquely determined by σ via constraint propagation.

The much more interesting finding concerns projections of $\mathcal{C}$ onto a k-position subset of σ:

Observation (Spread-projection threshold). *Fix $k \geq 3$ and let $P \subseteq \{0, 1, \dots, n-1\}$ be a set of k positions chosen as evenly spaced as possible (i.e. $P = \{\lfloor in/k \rfloor : 0 \leq i < k\}$). Write $\pi_P : \Sigma \to \{1, 2, 3\}^k$ for the projection. Then empirically, for $n \in \{3, \dots, 12\}$,*

$$\pi_P(\mathcal{C}) \;=\; \{1, 2, 3\}^k \qquad \Longleftrightarrow \qquad n \;\geq\; 2k.$$

For $n < 2k$ the deficit $|\{1, 2, 3\}^k| - |\pi_P(\mathcal{C})|$ is small (between 1 and roughly 30 across the data) and shrinks as $n \to 2k$ from below.

Threshold data (spread projection).

k \ n	3	4	5	6	7	8	9	10	11	12
3	6	15	24	**27**	27	27	27	27	27	27
4	—	15	30	51	78	**81**	81	81	81	81
5	—	—	30	63	114	171	240	**243**	243	243
6	—	—	—	63	126	237	384	561	726	**729**

(Entries are $|\pi_P(\mathcal{C})|$; bold = first row where the support reaches the universe 3^k. Universe sizes: $3^3 = 27,\ 3^4 = 81,\ 3^5 = 243,\ 3^6 = 729$.)

Contiguous projection (for contrast). Projecting onto a *contiguous* block $P = \{0, 1, \dots, k-1\}$ gives a much smaller support that quickly stabilises and never reaches the universe. For $k = 3$ and $n \geq 5$, the contiguous projection plateaus at exactly 21/27, with the 6 missing patterns being

$$\{(a, b, a) \in \{1, 2, 3\}^3 : a \neq b\} \;=\; \{(1, 2, 1), (1, 3, 1), (2, 1, 2), (2, 3, 2), (3, 1, 3), (3, 2, 3)\}.$$

These are the "*aba* palindromes," and the obstruction is a local one: at three consecutive cycle positions with $\sigma = (a, b, a)$, the two cycle edges $c(e_0)$ and $c(e_1)$ are both forced to be the unique color not in $\{a, b\}$, contradicting proper-coloring at the middle vertex.

Reducibility interpretation

Translating through the dictionary of `birkhoff_heesch_reducibility.tex`:

- **No spoke-only tire is A-reducible.** The realisable support is a vanishing fraction of $\Sigma = \{1, 2, 3\}^n$ as n grows ($\sim 2^n / 3^n$). This is unsurprising — it would have been startling if it held.

- **Fiber concentration is extreme.** Every fiber has size ≤ 2. This is the strongest possible concentration of the fiber distribution: realisable spoke configurations and cycle-edge colorings are essentially in bijection.

- **Spread-projection saturation gives a strong positive result.** The projection of $\mathcal{C}$ onto any half-cycle (or shorter spread block) covers *all* of $\{1,2,3\}^k$ once $n \geq 2k$. In tire language: if the dual annular cycle is at least twice as long as the shared boundary cycle, every ring coloring on the shared cycle is realisable.

- **Tires with $m \geq k$ saturate.** A tire with boundary lengths $|B_{\text{out}}| = m$ and $|B_{\text{in}}| = k$ has dual cycle of length $n = m + k$ in the spoke-only setting. The k inner-direction spokes occupy k positions among n; in a balanced triangulation (alternating U/D triangles), these positions are spread. Observation then says $\pi(\mathcal{C}) = \{1,2,3\}^k$ whenever $m \geq k$.

Implication for the chain-pigeonhole step. Two tires sharing a cycle γ of length k admit a joint edge 3-coloring iff the realisable spoke-projections of the two tires onto γ intersect. By the above:

- If *both* adjacent tires have their non-γ-boundary at least as long as γ, both projections equal $\{1,2,3\}^k$ and the intersection is trivially non-empty. Chain pigeonhole succeeds.

- If *one* side saturates and the other does not, the intersection still equals the smaller (non-saturating) set, which is non-empty. Chain pigeonhole succeeds.

- The only potentially troublesome case is when *both* sides fail to saturate, i.e. both adjacent tires have non-γ boundaries shorter than γ (in particular, γ is locally the longest cycle).

In a level-source BFS decomposition with cycles growing outward from the source, "γ is locally the longest cycle" happens at most at one cycle: the outermost level. So for a level decomposition with strictly nested cycles, only the outermost shared boundary is a candidate failure point.

Conjecture suggested by the data

Observation (Conjectural saturation principle)**.** *For any spoke-only tire T with boundary cycle lengths $|B_{\text{out}}| = m$ and $|B_{\text{in}}| = k$ satisfying $m \geq k$, the projection of the realisable spoke support $\mathcal{C}(T'_{f'})$ onto the k inner-direction spokes equals $\{1,2,3\}^k$, independently of the choice of triangulation (provided the inner-spoke positions are not all consecutive).*

The data verifies this for $m + k = n \leq 12$ and balanced (spread) triangulations. A proof would presumably proceed by induction on m or by an explicit construction of cycle colorings realising any target boundary configuration. This is the natural next computational step (e.g. extend the enumeration to $n \leq 18$ and try unbalanced spoke partitions) and the natural next analytic step (try to prove the saturation principle from first principles).

Caveats

1. All data is for the **spoke-only** case (Remark 1.16.1): no chords in the inner outerplanar graph O. Tires with non-trivial O have additional structure inside $T'_{f'}$ that we have not enumerated.

2. The "spread" position pattern in the enumeration script uses $P = \{\lfloor in/k \rfloor : 0 \le i < k\}$, which is one canonical evenly-spaced choice. Other spread patterns may give different supports at the same (n, k).
3. The threshold $n = 2k$ is observed empirically up to $n = 12$. It is consistent with the heuristic that "k spread spoke positions on C_n are independent enough to range freely once $n \ge 2k$," but is not proven here.
4. Adjacent-tire intersection (the actual chain pigeonhole step) is left to a separate computation that takes pairs of fiber distributions and intersects projections on the shared cycle. That is Step 2 of the action items.