Fiber decomposition of edge 3-colorings of tire annular face connectors

Why this note exists

The menagerie note gives a closed form (Section 6) for $P_e(G,k)$ when G is a 2-connected outerplanar graph with $\Delta \le 3$: a polygon C_n with a non-crossing chord matching M. But the object we actually want to count over, the *tire annular face connector* $T'_{f'}$ (Definition 1.16 of the main paper), is not that graph — it has extra pendant edges hanging off the cycle, one per non-chord boundary vertex. These pendants are the *spokes* (Definition 1.17).

This note explains the picture below: $T'_{f'}$ factors as a *core* (the polygon-with-matching $C_n+M = T'_{\mathrm{ann}}$ that the menagerie handles) plus a set of *spoke pendants* reaching into the neighbouring tires. Each edge 3-coloring of $T'_{f'}$ is the data of (i) a core edge coloring and (ii) a spoke coloring, and the spoke coloring is what compatibility-with-neighbours acts on.

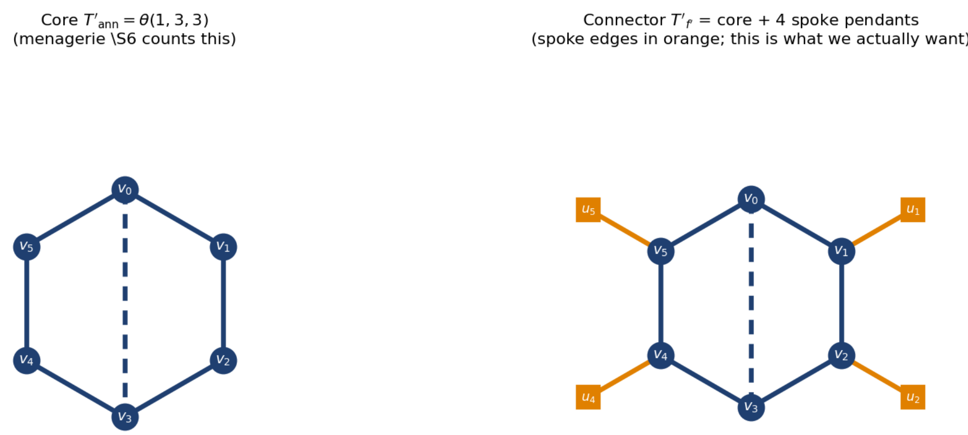

Figure 1: Anatomy of a tire annular face connector $T'_{f'}$ in the spoke-only case. **Left:** the core $T'_{\mathrm{ann}} = C_n + M$ (here $\theta(1,3,3)$), the object the menagerie §6 formula counts. **Right:** $T'_{f'}$ adds one pendant spoke edge for each non-chord boundary vertex; the spoke vertex is a dual vertex of a face of G *outside* the tire annulus. In a nested-tire setting, that outside face belongs to the next tire over.

Setup

Let $T \subseteq G$ be a tire with annular faces F_{ann}, let $T'_{\text{ann}} := G'[\{d_f : f \in F_{\text{ann}}\}]$ be the tire annular subgraph, fix a face f' of T'_{ann} with boundary walk $V(f')$, and let $T'_{f'}$ be the tire annular face connector (Definition 1.16). Write

$$S(f') \;:=\; V(T'_{f'}) \setminus V(f') \;=\; V_{\text{out}}(T'_{f'}) \sqcup V_{\text{in}}(T'_{f'})$$

for the set of *spokes* (outer and inner). In the spoke-only setting (Remark 1.16.1) each spoke $u \in S(f')$ contributes a single pendant edge vu to $T'_{f'}$ for a unique cycle vertex $v = v(u) \in V(f')$. The *spoke edge set* is

$$E_S \;:=\; \{\, v(u)\, u \;:\; u \in S(f') \,\} \;\subseteq\; E(T'_{f'}),$$

and $E(T'_{f'}) = E(T'_{\text{ann}}) \sqcup E_S$.

Spoke configurations and fibers

Definition (Spoke configuration)**.** *A* spoke configuration *on $T'_{f'}$ (with palette $[k]$) is a map $\sigma : E_S \to [k]$. Equivalently, σ assigns a color to the G'-edge from each spoke $u \in S(f')$ to its attachment vertex $v(u) \in V(f')$.*

Definition (Fiber)**.** *For $\sigma : E_S \to [k]$ the* fiber *of $T'_{f'}$ over σ is*

$$N(T'_{f'};\sigma) \;:=\; \#\{\, \textit{proper edge k-colorings of $T'_{f'}$ that restrict to σ on E_S} \,\}.$$

Trivially

$$\boxed{P_e(T'_{f'},\,k) \;=\; \sum_{\sigma : E_S \to [k]} N(T'_{f'};\,\sigma)}$$

and the support of the sum is the set of *realisable* spoke configurations.

What the fiber count really is. Fixing σ pins one extra color at each cycle vertex $v(u)$: the spoke edge $v(u)u$ contributes a forbidden color $\sigma(v(u)u)$ at $v(u)$. After this, the remaining problem is to proper-edge-color the core $T'_{\text{ann}} = C_n + M$ subject to vertex-level forbidden colors — precisely the constrained transfer matrix problem in the "General method" paragraph of menagerie §6, with the chord-color part of the forbidden-set assignment replaced by the spoke-color part.

If we write $C^{(v)}_\sigma \subseteq [k]$ for the set of forbidden colors at vertex v coming from spokes (so $|C^{(v)}_\sigma|$ is the number of spokes attached at v; in the spoke-only case it is 0 or 1), then

$$N(T'_{f'};\,\sigma) \;=\; \sum_{(c_1,\dots,c_r)\in[k]^r} N\big(C_n;\, \text{forbidden}_M(c_1,\dots,c_r) \cup C^{(\cdot)}_\sigma,\, k\big),$$

i.e. the menagerie sum but with the extra spoke-forbidden colors appended to the forbidden sets at each cycle vertex.

Special case: $E_S = \emptyset$. If there are no spokes (e.g. the spoke-free setting), σ is the empty map and the unique fiber is $N(T'_{f'};\,\sigma_\emptyset) = P_e(T'_{\text{ann}}, k)$, which is exactly the menagerie §6 formula.

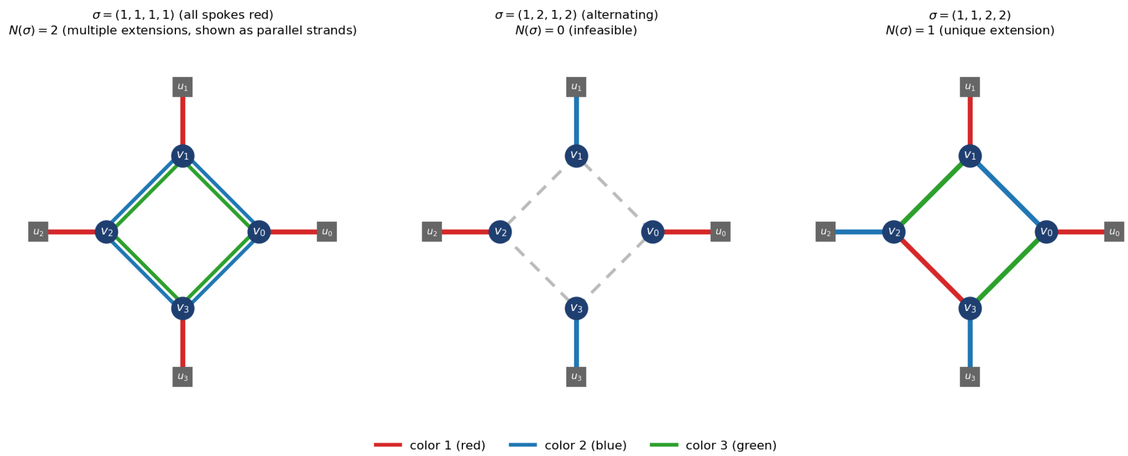

Figure 2: Three spoke configurations σ on C_4 with four pendant spokes, with cycle vertices labelled v_0, v_1, v_2, v_3 and spoke vertices u_0, u_1, u_2, u_3. **Left:** the constant configuration $\sigma = (1,1,1,1)$ (all spokes red) is realisable with fiber count $N(\sigma) = 2$ — the cycle alternates between the two remaining colors in two ways. **Middle:** the alternating configuration $\sigma = (1,2,1,2)$ is *infeasible* ($N(\sigma) = 0$); the constraints at v_0 and v_1 force $c(e_0) = 3$, but the constraint at v_2 then demands $c(e_1) \in \{2,3\} \setminus \{c(e_0)\}$ contradicting the v_1 constraint. **Right:** the non-uniform configuration $\sigma = (1,1,2,2)$ has fiber count $N(\sigma) = 1$ — a single cycle coloring extends it.

Small worked example: C_4 with four spokes at $k = 3$

Let $T'_{f'}$ be the 4-cycle $C_4 = v_0v_1v_2v_3$ with one pendant spoke v_iu_i at each cycle vertex. Each v_i has degree 3, so proper edge 3-coloring forces $\{c(s_i), c(e_{i-1}), c(e_i)\} = \{1,2,3\}$ at every v_i, where $s_i := v_iu_i$ and e_i is the cycle edge from v_i to v_{i+1} (indices mod 4).

Fix a spoke configuration $\sigma = (\sigma_0, \sigma_1, \sigma_2, \sigma_3)$. At each v_i, the two cycle edges e_{i-1}, e_i are the two colors of $[3] \setminus \{\sigma_i\}$ in some order. This gives the constraints

$$c(e_i) \in [3] \setminus \{\sigma_i, \sigma_{i+1}\}, \qquad i = 0,1,2,3,$$

together with $c(e_{i-1}) \neq c(e_i)$ at each v_i.

Total count. For C_4 alone, $P_e(C_4, 3) = P_v(C_4, 3) = 2^4 + 2 = 18$. At $k = 3$, each spoke is forced (the unique color avoiding its endpoint's two cycle-edge colors), so $P_e(T'_{f'}, 3) = 18$ as well: the pendants contribute a factor of 1^4.

Distribution over σ. The map "edge 3-coloring of $T'_{f'}$" $\mapsto$ "induced σ" is surjective onto its image, and

$$\sum_{\sigma \in [3]^4} N(T'_{f'}; \sigma) \;=\; 18.$$

The fiber sizes for the three configurations in Figure 2 can be checked directly from the constraints above; doing the same for every $\sigma \in [3]^4$ would partition the 18 colorings according to the spoke pattern they induce.

Why this matters for nested tires

The whole point of the fiber decomposition is that, in a nested-tire configuration, the spoke edge set of the inner tire is (a subset of) the spoke edge set of the outer tire *regarded from the other side.* Specifically, if two tires $T^{\text{outer}}, T^{\text{inner}}$ share a cycle γ in G, then for $v \in V(\gamma)$ the unique G'-edge at v that points out of γ's annular faces serves as an outer-spoke edge of one tire's face connector and an inner-spoke edge of the other's (Figure 3).

So an edge 3-coloring of the whole region decomposes as: a coloring of each tire's face connector, with the spoke configurations matched on the shared cycle. Writing σ_{shared} for the common spoke configuration on the shared boundary, this is

$$\#\{\text{joint colorings on } T^{\text{outer}} \cup T^{\text{inner}}\} \;=\; \sum_{\sigma_{\text{shared}}} N(T'^{\text{outer}}_{f'};\, \sigma_{\text{shared}}, *) \cdot N(T'^{\text{inner}}_{f'};\, \sigma_{\text{shared}}, *)$$

(with $*$ standing for the other-side spoke configuration, which is free). This is the conductivity step ϕ_B of the chain pigeonhole sketch from the conversation that prompted this note: the inner tire's profile, projected to its outer-side spoke configurations, must intersect the outer tire's profile, projected to its inner-side spoke configurations.

Where this leads. Two natural quantitative questions:

1. For a fixed face connector $T'_{f'}$ and a fixed boundary side, what is the distribution of $|N(T'_{f'};\, \sigma)|$ over σ? How concentrated is it, and how large is the realisable support?

2. For two nested tires to admit a joint edge 3-coloring it suffices that the supports of σ overlap. When can we guarantee this from the cycle length alone (which controls $|[k]^{|E_S|}|$ on the shared side)?

The fiber form makes both questions concrete: the first is a fiber distribution to compute, the second is a covering / pigeonhole on σ.

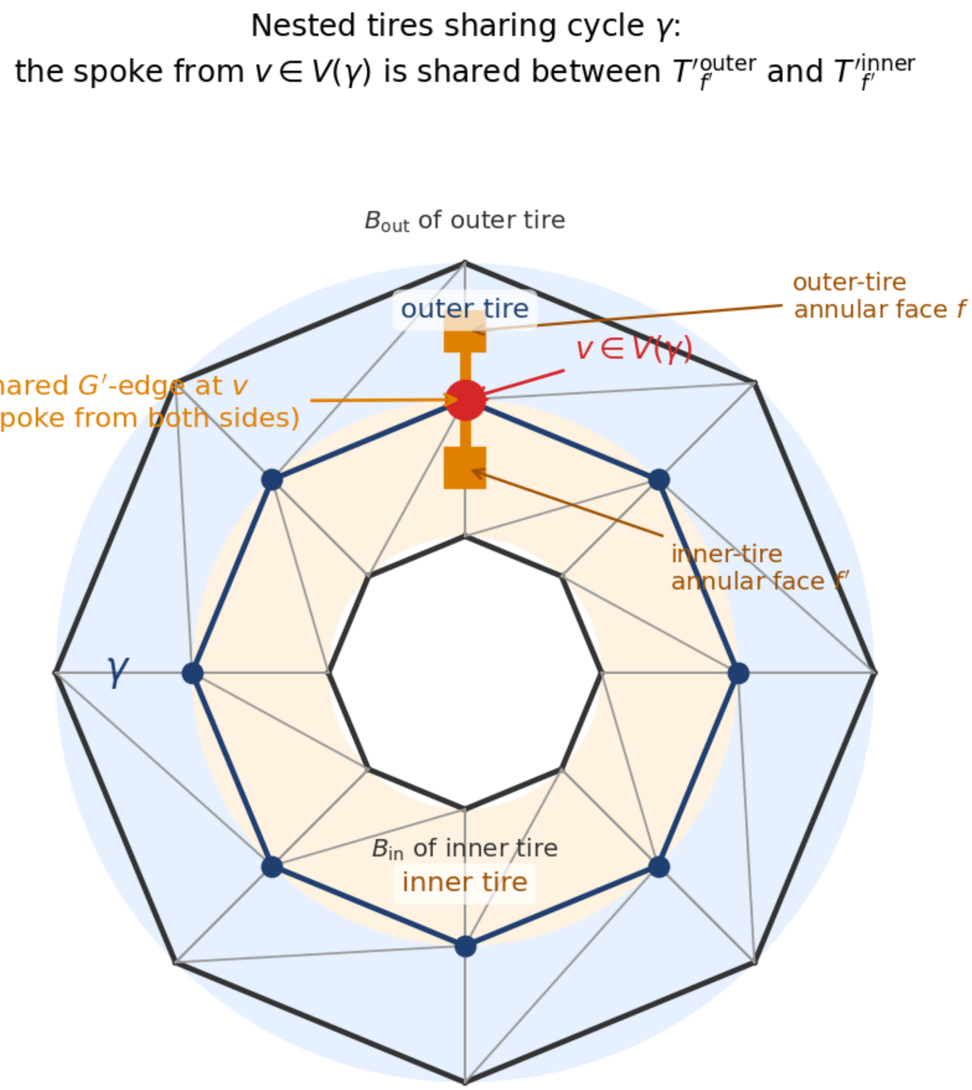

Figure 3: Two adjacent tires $T^{\text{outer}}, T^{\text{inner}}$ in a common maximal planar G, sharing the cycle γ. The cycle vertex $v \in V(\gamma)$ is a dual vertex for an annular face in *both* tires. The G'-edges at v split into: two cycle edges inside $T'^{\text{outer}}_{\text{ann}}$, two cycle edges inside $T'^{\text{inner}}_{\text{ann}}$, and (in the spoke-only case) *one shared spoke edge vu* pointing across the shared boundary. That single spoke edge is simultaneously a spoke of $T'^{\text{outer}}_{f'}$ and a spoke of $T'^{\text{inner}}_{f'}$, so any global edge coloring of G' assigns it a single color from both sides at once.