FLIP SYMMETRIC MAXIMAL PLANAR GRAPHS

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Abstract.

1. Motivation

The Four Color Theorem asserts that every planar graph is properly 4-colorable, or equivalently that no maximal planar graph G satisfies $\chi(G) \geq 5$. Suppose, towards a contradiction, that such a graph exists; let G_0 be one of minimum order. Any structural property shared by every maximal planar graph H with $|V(H)| = |V(G_0)|$ is then automatically inherited by G_0, and any property *not* satisfied by G_0 excludes a portion of the class of maximal planar graphs from playing the role of a minimum counterexample.

This paper investigates one such property: invariance under an admissible edge flip. We call a maximal planar graph G *flip-symmetric* when some admissible flip at an edge of G returns a graph isomorphic to G. Our principal observation (Theorem 4.1) is that a minimum-order 5-chromatic maximal planar graph cannot be flip-symmetric, so the search for a counterexample to the Four Color Theorem may, in principle, be confined to the complement of the class $\mathcal{F}$ of flip-symmetric graphs. This raises a quantitative question — how large is $\mathcal{F}$? — which we address empirically in Section 5 by an exhaustive census of maximal planar graphs of small order.

2. Preliminaries

Let G be a maximal planar graph with $|V(G)| \geq 4$, embedded in the plane so that every face — including the outer face — is a triangle. Every edge $uv \in E(G)$ is then shared by exactly two triangular faces uvw and uvx whose union is a quadrilateral $uwvx$ with diagonal uv.

Definition 2.1 (Edge flip)**.** Let G be a maximal planar graph and let $uv \in E(G)$ be an edge whose two incident triangular faces are uvw and uvx. The *edge flip* (or *diagonal flip*) at uv is the operation that deletes the edge uv and inserts the edge wx in its place, replacing the two triangles uvw and uvx by the two triangles uwx and vwx. The flip is *admissible* if $wx \notin E(G)$; otherwise the resulting multigraph is not simple and the flip is forbidden.

3. Flip-symmetric maximal planar graphs

For a maximal planar graph G and an admissible edge $uv \in E(G)$ with incident triangles uvw, uvx, write

$$G^{\mathrm{flip}(uv)} = \big(V(G),\, (E(G) \setminus \{uv\}) \cup \{wx\}\big)$$

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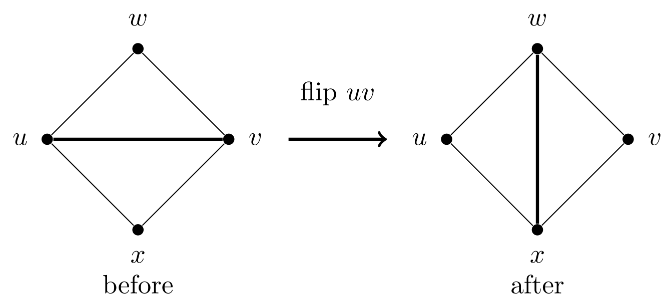

FIGURE 1. An edge flip replaces the diagonal uv of the quadrilateral $uwvx$ with the diagonal wx.

for the graph obtained from G by flipping uv.

Definition 3.1 (Flip-symmetric graph)**.** A maximal planar graph G is *flip-symmetric* if there exists an admissible edge $uv \in E(G)$ such that $G^{\mathrm{flip}(uv)} \cong G$. We write $\mathcal{F}$ for the class of flip-symmetric maximal planar graphs.

4. A minimal four-colorable counterexample

Theorem 4.1. *Let G be a maximal planar graph of minimum order among all maximal planar graphs H with $\chi(H) \geq 5$. Then $G \notin \mathcal{F}$; that is, G is not flip-symmetric.*

5. Flip symmetry frequency

To gauge how restrictive flip-symmetry is, we performed an exhaustive census of maximal planar graphs of small order. For each $n \in \{4, 5, \ldots, 14\}$ we enumerated every isomorphism class of maximal planar graph on n vertices using `plantri` (invoked through SageMath as `graphs.planar_graphs` with `minimum_connectivity` = 3 and `maximum_face_size` = 3), and for each such G we tested every admissible edge $uv \in E(G)$ for the existence of an isomorphism $G \cong G^{\mathrm{flip}(uv)}$. Writing T_n for the total number of maximal planar graphs on n vertices and $F_n = |\mathcal{F} \cap \{G : |V(G)| = n\}|$ for the number of flip-symmetric ones, the results are tabulated below.

n	T_n	F_n	F_n/T_n
4	1	0	0.000000
5	1	1	1.000000
6	2	1	0.500000
7	5	1	0.200000
8	14	5	0.357143
9	50	17	0.340000
10	233	48	0.206009
11	1,249	164	0.131305
12	7,595	552	0.072679
13	49,566	1,828	0.036880
14	339,722	6,164	0.018144

From $n = 10$ onward the ratio F_n/T_n decreases by a factor approaching $1/2$ at each step, suggesting that the density of flip-symmetric graphs among maximal planar graphs of order n decays to zero — empirically at a roughly geometric rate. This tempers the utility of Theorem 4.1: although it guarantees that a minimum-order counterexample to the Four Color Theorem lies in the complement of $\mathcal{F}$, that complement already comprises nearly the entire class of maximal planar graphs on n vertices once n is moderately large. The structural exclusion offered by flip-symmetry therefore prunes a vanishingly small portion of the search space, and this property is unlikely on its own to be a productive avenue for narrowing the search for a counterexample.

A natural follow-up question is whether the picture improves when one restricts attention to maximal planar graphs of minimum degree at least 5, the class to which any minimum-order 5-chromatic graph necessarily belongs (a vertex of degree at most 4 admits a standard Kempe reduction). Writing $T_n^{(5)}$ and $F_n^{(5)}$ for the analogous counts within this subclass, we ran the same census after adding `minimum_degree` $= 5$ to the `plantri` invocation, obtaining the table below.

n	$T_n^{(5)}$	$F_n^{(5)}$	$F_n^{(5)}/T_n^{(5)}$
12	1	0	0.000000
13	0	0	—
14	1	0	0.000000
15	1	0	0.000000
16	3	1	0.333333
17	4	1	0.250000
18	12	2	0.166667
19	23	5	0.217391
20	73	12	0.164384
21	192	27	0.140625
22	651	51	0.078341
23	2,070	120	0.057971
24	7,290	273	0.037449
25	25,381	598	0.023561
26	91,441	1,341	0.014665

The first flip-symmetric example in this subclass appears at $n = 16$. Beyond that, the density $F_n^{(5)}/T_n^{(5)}$ again decays toward zero, though at a noticeably gentler rate: the step-to-step ratio settles around 0.63 rather than the $\approx 1/2$ observed in the unrestricted census. The restriction to minimum degree 5 therefore preserves flip-symmetry slightly longer relative to the size of the subclass, but does not alter the qualitative conclusion: even within the minimum-degree-5 class — which already contains every candidate minimum-order 5-chromatic graph — flip-symmetric examples become a vanishing fraction.

6. Further necessary properties of a minimal counterexample

The frequency data of Section 5 look unflattering only when flip-symmetry is weighed against the full class of maximal planar graphs. The class that actually matters — minimum-order 5-chromatic triangulations that also resist every Kempe-style reduction — is far thinner, and flip-symmetry may exclude a substantially larger fraction of it if the configurations it removes overlap those responsible for Kempe reducibility. We therefore turn to identifying further necessary properties of a minimum-order 5-chromatic maximal planar graph, of which flip-asymmetry is the first.

7. Edge-deletion subgraphs

Definition 7.1 (Edge-deletion subgraph)**.** Let G be a maximal planar graph and $uv \in E(G)$. The *edge-deletion subgraph at uv* is the spanning subgraph $G - uv = (V(G),\, E(G) \setminus \{uv\})$. Write $\mathcal{D}(G) = \{G - uv : uv \in E(G)\}$.

Theorem 7.2. *Let G_0 be a maximal planar graph of minimum order with $\chi(G_0) \geq 5$. Then every $H \in \mathcal{D}(G_0)$ is 4-colorable.*

Proof. Fix $uv \in E(G_0)$ and let G_0/uv denote the simple planar graph obtained by contracting uv and discarding parallel edges. Since $|V(G_0/uv)| = |V(G_0)| - 1$, the minimality of G_0 supplies a proper 4-coloring c of G_0/uv. Let z be the contracted vertex and define $c' \colon V(G_0) \to \{1,2,3,4\}$ by $c'(u) = c'(v) = c(z)$ and $c'(y) = c(y)$ for $y \notin \{u,v\}$. Every edge of $G_0 - uv$ is either disjoint from $\{u,v\}$ or incident to exactly one of them; in either case the corresponding edge of G_0/uv has distinct endpoints under c, so c' assigns its endpoints distinct colors. The edge uv itself is absent from $G_0 - uv$, so c' is a proper 4-coloring of $G_0 - uv$. $\square$