Edge 3-colorings of small outerplanar graphs with $\Delta \le 3$: a menagerie

Setup

For a graph G write $P_e(G,k)$ for the number of proper k-edge-colorings of G (= the chromatic polynomial of the line graph $L(G)$ evaluated at k). Throughout this note $\Delta(G)$ denotes the maximum degree of G; we are interested in $k = 3$ and $\Delta(G) \le 3$, with G outerplanar.

There is no *universal* closed form for $P_e(G,3)$ on the class of subcubic outerplanar graphs, but the class is small enough that every G in it decomposes along its block–cut tree into building blocks each of which admits a closed-form count. The building blocks form a short menagerie.

The menagerie

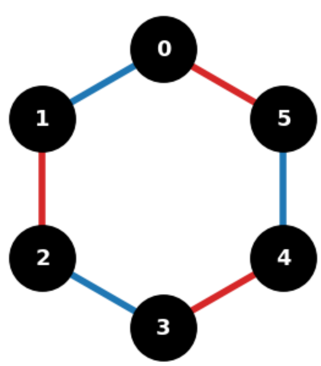
1. Path P_n (n vertices, $n-1$ edges; $\Delta \le 2$)

The line graph $L(P_n)$ is the path P_{n-1}, so

$$P_e(P_n,k) \;=\; P_{\text{vert}}(P_{n-1},k) \;=\; k(k-1)^{n-2}, \qquad P_e(P_n,3) \;=\; 3\cdot 2^{n-2}.$$

(For a single edge, $n = 2$, the count is 3; for two edges in a row, $n = 3$, the count is 6.)

2. Cycle C_n ($\Delta = 2$)

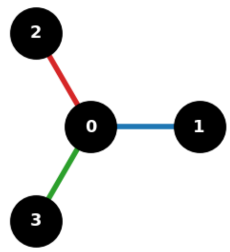


The line graph $L(C_n) = C_n$, so

$$P_e(C_n, k) \;=\; (k-1)^n + (-1)^n(k-1), \qquad P_e(C_n, 3) \;=\; 2^n + 2(-1)^n.$$

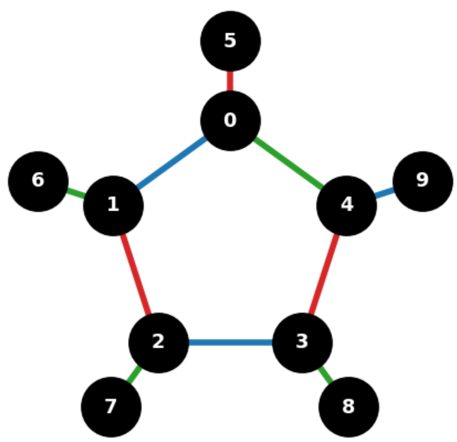
For even n the count is $2^n + 2$; for odd n it is $2^n - 2$.

3. Star $K_{1,3}$ (a single $\Delta = 3$ vertex)



Three pairwise-incident edges at a single vertex must carry three distinct colors, so $P_e(K_{1,3}, 3) = 3! = 6$. More generally $P_e(K_{1,d}, k) = k(k-1)\cdots(k-d+1)$, which is positive iff $k \geq d$, i.e. iff $k \geq \Delta(G)$.

4. Corona $C_n \circ K_1$ (cycle with one leaf per cycle vertex; $\Delta = 3$)



Each cycle vertex v has degree 3 in $C_n \circ K_1$: its two cycle edges must carry distinct colors and its leaf must carry the unique remaining third color. So the leaf coloring is *forced* by the cycle coloring, and

$$P_e(C_n \circ K_1, 3) \;=\; P_e(C_n, 3) \;=\; 2^n + 2(-1)^n.$$

This is the form of the partial tire dual $D(T)$ in the spoke-only case (with $L = n + m$).

5. Trees with $\Delta \leq 3$

A tree T on n vertices has $|E(T)| = n - 1$ edges; its line graph $L(T)$ is a *block graph* (every block is a clique). Edge-color a tree greedily by processing edges in BFS order from a leaf: when an edge $\{u, v\}$ is added, the only colors forbidden are those already used on the edges incident to its already-colored endpoint. Hence at any vertex of degree d, when the d-th edge is added there are exactly $k - (d - 1)$ choices. For $k = 3$ and $\Delta \leq 3$:

$$P_e(T, 3) \;=\; 3 \prod_{e \in E(T) \setminus \{e_0\}} (3 - d_e),$$

where e_0 is the first edge processed and d_e is the number of already- processed edges incident to the new endpoint of e (between 1 and 2, since $\Delta \leq 3$). In practice this gives a clean product depending only on the degree sequence of T.

6. Two-connected outerplanar with $\Delta \leq 3$: cycle, possibly with a matching of chords

The 2-connected outerplanar graphs are polygons (with optional chords). Each chord raises the degree of its two endpoints by 1, and a polygon vertex already has degree 2 from the cycle. So for $\Delta \leq 3$ the chords must form a *matching* of polygon vertices — no vertex can be an endpoint of two chords. In particular the simplest non-cycle case is a polygon with a *single* chord, denoted $\theta(1, p, q)$: two paths of lengths p and q between two trivalent vertices, plus a direct edge between those two trivalent vertices. Equivalently it is C_{p+q} with a chord joining the two cycle vertices at distance p apart on the polygon.

[$\theta(1, p, q)$: polygon C_{p+q} with one chord; both chord endpoints have degree 3, other polygon vertices have degree 2.]

$\theta(1, p, q)$ is outerplanar (the chord lies inside the polygon, and all polygon vertices are on the outer face) and subcubic.

Closed form. For each choice of chord color $c \in \{1, 2, 3\}$, the two cycle edges incident to each chord endpoint must lie in $\{x, y\} = \{1, 2, 3\} \setminus \{c\}$. Conditioning on whether the two pairs of cycle-edge colors at the chord endpoints agree or disagree, and using the transfer matrix $T = J - I$ on K_3 (eigenvalues 2 and -1 with multiplicities 1 and 2), one finds

$$\boxed{P_e(\theta(1, p, q),\, 3) \;=\; \frac{2^{p+q} - 2^p(-1)^q - 2^q(-1)^p + 10\,(-1)^{p+q}}{3}}.$$

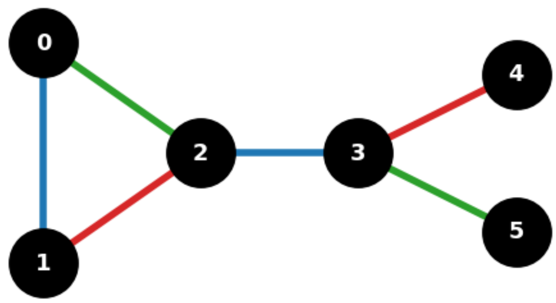
Sanity checks:

- $p = q = 2$: $\theta(1, 2, 2) = K_4 - e$, formula gives $(16 - 4 - 4 + 10)/3 = 6 = P_e(K_4 - e, 3)$.
- $p = q = 3$: $\theta(1, 3, 3)$, formula gives $(64 + 8 + 8 + 10)/3 = 30$.
- $p = q = 6$: $\theta(1, 6, 6)$ (= the interior dual subgraph of the partial tire dual for the barbell-O tire of Figure 4 of the main paper), formula gives $(4096 - 64 - 64 + 10)/3 = 1326$.

The formula has been verified empirically against `Sage`'s chromatic polynomial routine for all $p, q \in \{2, 3, 4, 5, 6\}$.

More generally, a polygon with r chords forming a matching has chromatic polynomial computable by the same transfer-matrix idea along the $r+1$ paths between consecutive chord endpoints on the polygon, with a product constraint at each chord endpoint.

7. Block–cut decomposition

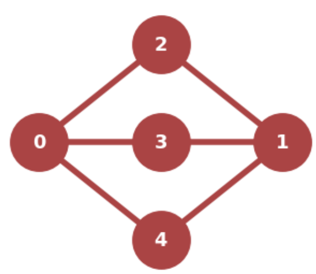


A general subcubic outerplanar graph is a union of cycle-blocks and edge-blocks glued at cut vertices. By case 6 each 2-connected block is a cycle; the remaining blocks are single edges (i.e. tree edges). At a cut vertex v of degree $d_v \in \{2, 3\}$, the colors of the d_v edges incident to v must be pairwise distinct. Counting $P_e(G, 3)$ for the whole graph G amounts to counting colorings of each block independently and then enforcing the distinct-color constraint at every cut vertex. For $k = 3$ and $\Delta \leq 3$ this gives

$$P_e(G,3) \;=\; \prod_{B \text{ cycle block of } G} P_e(B,3) \;\cdot\; \prod_{B \text{ edge block of } G} P_e(B,3) \;/\; \prod_{v \text{ cut vertex}} (\text{normalization at } v),$$

where the normalization corrects the over- or under-counting at the cut-vertex constraint. For each cycle-block $B = C_n$ contributing $2^n + 2(-1)^n$ proper 3-edge-colorings, and each edge-block contributing 3, this product is computable in time linear in $|V(G)| + |E(G)|$.

Outside the menagerie: $\theta(p, q, r)$ with all paths ≥ 2, i.e. $K_{2,3}$ subdivisions



The genuine theta graph $\theta(2, 2, 2) = K_{2,3}$ (and more generally any $\theta(p, q, r)$ with $p, q, r \geq 2$) is *not* outerplanar: $K_{2,3}$ is a forbidden minor for outerplanarity, and every $\theta(p, q, r)$ with $\min(p, q, r) \geq 2$ contains $K_{2,3}$ as a topological minor. Such graphs do *not* arise as the interior dual subgraph of a partial tire dual $D(T)$ in this paper: when the inner outerplanar graph O has a bridge, the interior dual subgraph is the polygon-with-one-chord case $\theta(1, p, q)$ of §6, where the path of length 1 is precisely the dual of the bridge. In particular $D(T)$ *is* subcubic outerplanar in the bridge case, and its edge 3-coloring count is the $\theta(1, p, q)$ closed form above.