COLORING NESTED TIRE GRAPHS

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Abstract.

1. Introduction

A classical theorem of Tait recasts the Four Colour Theorem in dual, edge-colouring terms: a plane triangulation G is properly 4-vertex-colourable if and only if its dual cubic graph G' is properly 3-edge-colourable. Thus a minimal counterexample to the Four Colour Theorem – a smallest triangulation admitting no proper 4-colouring – corresponds to a smallest cubic plane graph admitting no proper 3-edge-colouring.

We study the structure such a minimal counterexample would have to exhibit through the lens of *nested level duals.* Fixing a level source S in G endows the dual G' with a Breadth-First-Search–derived labelling, the dual depth of Definition 1.4, and the level structure of G organises G' into a family of nested cycles carrying these labels. Our aim is to express the obstruction to a 3-edge-colouring of G' as conditions on this nested labelled-cycle structure.

Throughout, $G = (V, E)$ is a plane maximal planar graph (a triangulation) with a fixed planar embedding Π_G. We write $|V| = n$, so $|E| = 3n - 6$ and G has $2n - 4$ triangular faces.

Definition 1.1 (Level source)**.** A *level source* of G is any vertex $v \in V$; we write $S = \{v\}$ for the level-0 source.

Definition 1.2 (Levels)**.** Given a level source $S \subseteq V$, the *level* of $v \in V$ is $\ell_G(v) = \mathrm{dist}_G(v, S)$, the graph distance from v to the nearest source vertex.

Definition 1.3 (Dual)**.** The *dual* of G, written G', is the inner (weak) planar dual of G with respect to the embedding Π_G: it has one vertex d_f for each bounded face f of G, and an edge joining d_f and $d_{f'}$ for each edge of G shared by two bounded faces f and f'. The unbounded outer face contributes no vertex, and edges of G on the outer boundary contribute no dual edge. Since G is a triangulation, each vertex $d_f \in V(G')$ corresponds to a triangular face f of G, and we write $V(f) \subseteq V$ for its three incident vertices.

Definition 1.4 (Dual depth)**.** Given a level source $S \subseteq V$, the *dual depth* of a dual vertex $d_f \in V(G')$ is

$$\delta_G(d_f) = \min_{v \in V(f)} \ell_G(v) = \min_{v \in V(f)} \mathrm{dist}_G(v, S),$$

the smallest level among the three vertices of G bounding the face f.

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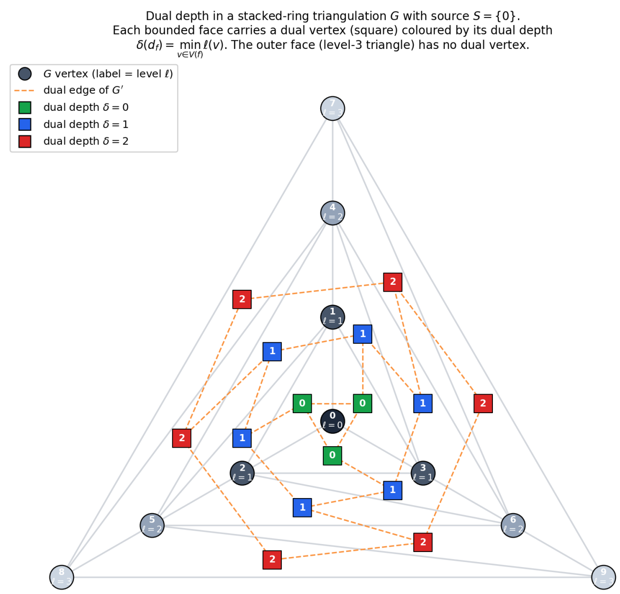

FIGURE 1. Dual depth in a stacked-ring triangulation G with level source $S = \{0\}$. Each G vertex is labelled by its level ℓ. Each bounded face carries a dual vertex (square, joined by dashed dual edges) coloured by its dual depth $\delta(d_f) = \min_{v \in V(f)} \ell(v)$: the central fan has depth 0, the inner annulus depth 1, and the outer annulus depth 2. The outer face (the level-3 triangle) is excluded from the inner dual and carries no dual vertex.

Definition 1.5 (Tire graph)**.** A *tire graph* consists of a plane graph T together with an *outer boundary* $B_{\text{out}} \subseteq T$ and an *inner outerplanar graph* $O \subseteq T$ with $V(B_{\text{out}}) \cap V(O) = \emptyset$, where

- B_{out} is either a simple cycle of length ≥ 3 or a single vertex (a *degenerate outer boundary*);
- O is an outerplanar graph; its *inner boundary* B_{in} is the closed walk in O that traces the boundary of O's outer face in the inherited embedding, which is a simple cycle when O is 2-connected and a non-simple closed walk in general (visiting bridges twice and cut-vertices multiple times); if $|V(O)| = 1$, we say T has a *degenerate inner boundary*.

At most one of $B_{\text{out}}, B_{\text{in}}$ may be degenerate. The vertex and edge sets of T are

$$V(T) = V(B_{\text{out}}) \cup V(O), \qquad E(T) = E(B_{\text{out}}) \cup E(O) \cup E_{\text{ann}},$$

where E_{ann} — the *annular edges* — has the property that, in the plane embedding of T, the closed planar region R bounded externally by B_{out} and internally by B_{in} is partitioned into triangular faces of T whose union is R.

When B_{out} is a simple cycle and O is 2-connected, R is a closed annulus. More generally, R is a closed planar region that may fail to be a 2-manifold at cut-vertices of O (where two "lobes" of the depth-d region meet at a single vertex); the inner boundary B_{in} is then a non-simple closed walk that visits the cut-vertex multiple times. The relaxed definition accommodates outerplanar inner graphs with bridges, cut-vertices, or multiple connected components. When either boundary is degenerate, R is a closed disk with that vertex as apex.

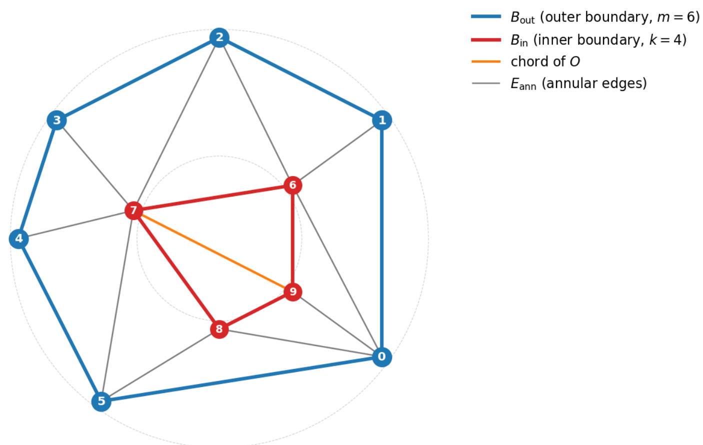

FIGURE 2. A tire graph with non-degenerate boundaries: outer boundary B_{out} a 6-cycle on vertices $0, \dots, 5$ (blue), inner boundary B_{in} a 4-cycle on vertices $6, \dots, 9$ (red), inner outerplanar graph $O = B_{\text{in}} \cup \{7\text{–}9\}$ (with one chord, orange), and E_{ann} (grey) tiling the annulus between B_{out} and B_{in} by ten triangular faces.

Remark 1.6. Let $m = |V(B_{\text{out}})|$ and $k = |V(B_{\text{in}})|$. By Euler's formula on the annular (resp. disk) region R, the tire graph has $m+k$ triangular faces inside R and $|E_{\text{ann}}| = m + k$ annular edges when neither boundary is degenerate; when exactly one boundary is degenerate (so $\min(m, k) = 1$), there are $m+k-1$ triangular faces and $|E_{\text{ann}}| = m + k - 1$.

Definition 1.7 (Partial tire dual)**.** Let $T = (B_{\text{out}}, O, E_{\text{ann}})$ be a tire graph in the sense of Definition 1.5, and let F_{ann} denote the set of triangular faces of T in the closed annular region between B_{out} and B_{in}. The *partial tire dual* of T, written $D(T)$, is the graph defined as follows.

Vertices.

(V1) For each face $f \in F_{\text{ann}}$, an *interior vertex* d_f of $D(T)$.
(V2) For each edge $e \in E(B_{\text{out}})$, a *leaf vertex* ℓ_e^{out}.
(V3) For each occurrence of an edge in the closed walk B_{in} (= the outer-face boundary walk of O), a *leaf vertex* ℓ_e^{in}. (When O is 2-connected each edge appears once; cut-vertices and bridges of O may cause an edge or vertex to appear more than once.)

Edges.

(E1) For each edge $e \in E(T)$ whose two incident faces both lie in F_{ann} (an *interior annular edge*), one edge $\{d_{f_1}, d_{f_2}\} \in E(D(T))$ where $f_1, f_2 \in F_{\text{ann}}$ are the two annular faces incident to e.

(E2) For each $e \in E(B_{\text{out}})$, one edge $\{d_f, \ell_e^{\text{out}}\} \in E(D(T))$ where $f \in F_{\text{ann}}$ is the unique annular face incident to e. The leaf ℓ_e^{out} has degree 1.

(E3) For each occurrence of e on the boundary walk B_{in}, one edge $\{d_f, \ell_e^{\text{in}}\} \in E(D(T))$ where $f \in F_{\text{ann}}$ is the annular face incident to e on the side of that occurrence. The leaf ℓ_e^{in} has degree 1.

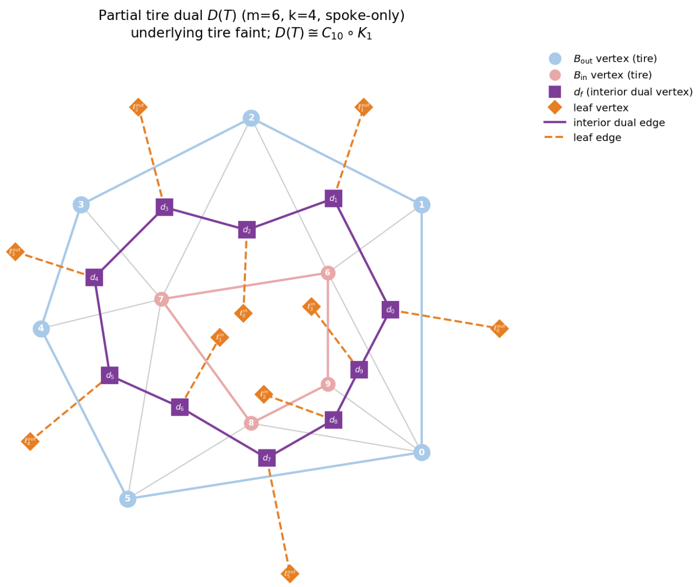

Figure 3. The partial tire dual $D(T)$ (purple squares + orange diamonds) drawn on top of a small tire graph T (faint) with $m = 6$ and $k = 4$. The ten interior vertices d_f at the centroids of the annular triangles form a single 10-cycle (solid purple); each boundary edge of the annular region (either of B_{out} or of B_{in}) contributes a degree-1 leaf (orange diamond) attached to the unique annular face incident to it (dashed orange), giving the structure $C_{10} \circ K_1$ of Proposition 1.8.

Proposition 1.8 (Structure of $D(T)$ when the annular triangulation is spoke-only)**.** *Suppose B_{out} is a simple cycle of length n, O is a 2-connected outerplanar graph whose outer-face cycle B_{in} has length m, and E_{ann} consists only of* spokes *(edges with one endpoint in $V(B_{\text{out}})$ and one in $V(B_{\text{in}})$). Then each face $f \in F_{\text{ann}}$ has exactly one boundary edge (on B_{out} or B_{in}) and two interior annular edges, and consequently $D(T)$ is isomorphic to the corona graph $C_{n+m} \circ K_1$: a cycle of length $n + m$ on the interior vertices $\{d_f\}$, with one leaf attached to each cycle vertex.*

In particular, $|V(D(T))| = 2(n + m)$ and $|E(D(T))| = 2(n + m)$.

Proof. Each annular triangle f in a spoke-only triangulation has the form $\{x, y, z\}$ with $x \in V(B_{\text{out}})$, $y \in V(B_{\text{in}})$, and z also in $V(B_{\text{out}}) \cup V(B_{\text{in}})$. Of its three edges, the one between the two same-side vertices (x-z if both on B_{out}, or y-z if both on B_{in}) is a boundary edge of the annular region; the other two edges are spokes.

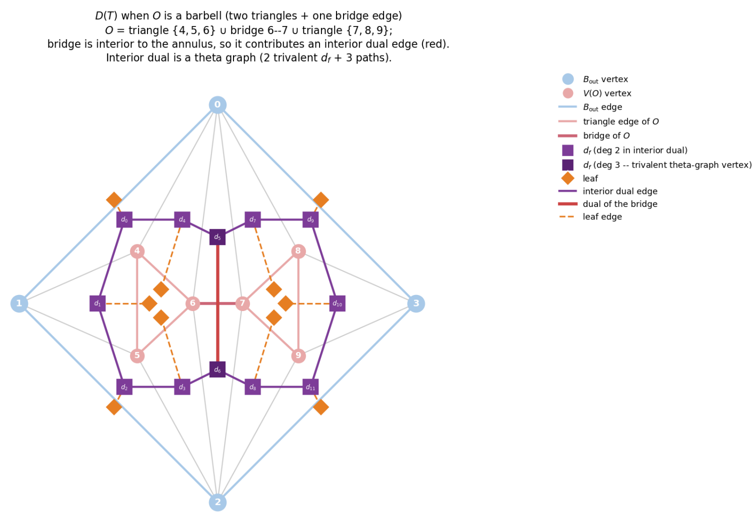

FIGURE 4. Partial tire dual $D(T)$ when the inner outerplanar graph O has a bridge — here a non-trivial edge cut connecting two disjoint triangles. B_{out} is a 4-cycle on $\{0, 1, 2, 3\}$ and O is the barbell: triangle $\{4, 5, 6\}$ together with triangle $\{7, 8, 9\}$ joined by the bridge edge 6–7 (removing the bridge disconnects O). Because both faces incident to the bridge are annular triangles, the bridge contributes an *interior dual edge* (highlighted in red) rather than two leaves; consequently the interior dual subgraph is no longer the single $(n+m)$-cycle of Proposition 1.8, but a theta graph: the two trivalent vertices d_5, d_6 (the bridge-incident annular faces) are joined by three internally vertex-disjoint paths in $D(T)$. Leaves come only from B_{out} ($n = 4$ leaves) and the six non-bridge edges of O ($m_\partial = 6$ leaves, three for each triangle).

So each d_f has degree 3 in $D(T)$: two from interior edges (= spokes shared with adjacent annular faces) and one leaf. The induced subgraph on $\{d_f : f \in F_{\text{ann}}\}$ is 2-regular; together with the connectedness of the annular region this forces it to be a single cycle. By Remark 1.6, the cycle has length $n+m$, and there are also $n+m$ leaves attached one-per-cycle-vertex. $\square$

Proposition 1.9 (Source-side simple-cycle property)**.** *Let G be a maximal planar graph with planar embedding Π_G and single-vertex source v_0. Let $d \geq 1$, $v \in L_d$, and let C' be a connected component of G'_d such that v is incident to some face in $F_{C'}$. Then the depth-d faces in $F_{C'}$ incident to v form a single contiguous arc in v's rotation in Π_G.*

Equivalently: for any such component, the source-side boundary of $R_{C'}$ is a simple cycle in L_d (no cut-vertices at level d).

Proof. Suppose for contradiction that the depth-d faces in $F_{C'}$ at v lie in two or more disjoint arcs of v's rotation. Adjacent vertices in G differ in level by at most 1, so a face at v has depth exactly d iff both other vertices have level $\geq d$, and depth $\leq d-1$ iff at least one has level $d-1$. Hence the gaps between the depth-d arcs at v are populated by level-$(d-1)$ neighbours of v, occurring in at least two disjoint arcs of v's rotation. Pick p in one such gap and q in another.

The BFS ball $G[L_{<d}]$ is connected, so there exists a simple path P in $G[L_{<d}]$ from p to q. Define the closed walk

$$W \;:=\; v \to p \to P \to q \to v.$$

Every vertex of P lies in $L_{<d}$, while $\ell(v) = d$, so v is distinct from every vertex of P; P is simple, so its internal vertices are distinct; and $p \neq q$ since they lie in different gaps. Hence W is a simple cycle in G.

By the Jordan curve theorem, the planar embedding of W divides Π_G into two regions. In v's rotation, the edges $v-p$ and $v-q$ lie at two specific positions, and they split the rotation into two arcs; each arc lies in one of the two regions determined by W. By choice of p, q, the two arcs of depth-d faces at v in $F_{C'}$ lie in different regions of W (i.e., one arc on each side).

Since C' is connected in G' and contains depth-d faces in both arcs, there is a dual path $f_1, f_2, \ldots, f_k$ in G'_d with $f_1, f_k \in F_{C'}$ incident to v in different arcs, and with the intermediate faces $f_2, \ldots, f_{k-1}$ not incident to v (a shortest such dual path). Consecutive faces f_i, f_{i+1} share an edge e_i of G; for $i \geq 2$, both endpoints of e_i lie in $L_{\geq d}$ (since neither f_i nor f_{i+1} is incident to v, all six vertices of these two triangles lie in $L_{\geq d}$). In particular, e_i shares no endpoint with W except possibly v — and v is excluded from $f_2, \ldots, f_{k-1}$.

A planar edge with neither endpoint on a simple closed planar curve W has both of its incident faces on the same side of W. Applying this to each e_i ($i \geq 2$) inductively: starting from f_2 on the same side of W as f_1 (their shared edge $e_1 = w - w'$ opposite to v in f_1 has $w, w' \in L_{\geq d}$ and hence is not on W), the path $f_2 \to f_3 \to \cdots \to f_{k-1} \to f_k$ stays on one side of W.

But f_1 and f_k lie on different sides of W (by construction), contradicting the conclusion that the entire path lies on one side. $\square$

Lemma 1.10 (Tire-component lemma)**.** *Let G be a maximal planar graph and let $S \subseteq V(G)$ be a level source. Fix a plane embedding Π_G of G in which S lies on the outer face (such an embedding exists for any planar graph and any single-vertex source). For $d \geq 0$, let*

$$G'_d \;:=\; G'\big[\,\{d_f \in V(G') : \delta_G(d_f) = d\}\,\big]$$

be the inner-dual subgraph on dual vertices of dual depth d, and let C' be a connected component of G'_d. Write $F_{C'} := \{f : d_f \in V(C')\}$, $V_{C'} := \bigcup_{f \in F_{C'}} V(f)$, and let $C := G[V_{C'}]$ inherit its embedding from Π_G. Set $R_{C'} := \bigcup_{f \in F_{C'}} f \subseteq |\Pi_G|$.

Then C, with the inherited embedding, is a tire graph in the sense of Definition 1.5. Its outer boundary B_{out} is the side of $R_{C'}$ closer to S in Π_G, namely the level-d subgraph $G[V_{C'} \cap L_d]$ (a simple cycle or single vertex); its inner outerplanar graph is $O = G[V_{C'} \cap L_{d+1}]$, and its inner boundary B_{in} is the outer-face boundary closed walk of O in the inherited embedding (a simple cycle when O is 2-connected, a non-simple closed walk in general). The triangular faces of C inside the closed boundary region are exactly the faces of G in $F_{C'}$.

Proof. Outerplanarity of the two level parts. By construction S lies on the outer face of Π_G, so Lemma 2.6 of [1] applies directly with (G, Π_G, S), giving that $G[L_{d'}]$

is outerplanar for each $d' \geq 0$. Subgraphs of outerplanar graphs are outerplanar, so $G[V_{C'} \cap L_d]$ and $G[V_{C'} \cap L_{d+1}]$ are both outerplanar.

Layer containment. Each $f \in F_{C'}$ has at least one vertex at level d, and adjacent vertices in G differ in level by at most 1; combined with $\delta_G(d_f) = d$, this forces $V(f) \subseteq L_d \cup L_{d+1}$. Hence $V_{C'} \subseteq L_d \cup L_{d+1}$, and C has vertex partition $V_{C'} = (V_{C'} \cap L_d) \sqcup (V_{C'} \cap L_{d+1})$.

Boundary edges are monochromatic in level. Each edge e on $\partial R_{C'}$ separates a face $f \in F_{C'}$ from a face $f' \notin F_{C'}$. Because f and f' share the edge e, their dual vertices are adjacent in G'; if both had depth d they would lie in the same component of G'_d, contradicting $d_f \in C'$ and $d_{f'} \notin C'$. Hence $\delta_G(d_{f'}) \neq d$; combined with the bounded-step property of δ across G'-adjacent faces, $\delta_G(d_{f'}) \in \{d-1, d+1\}$.

- If $\delta_G(d_{f'}) = d-1$, the third vertex w of $f' = \{u, v, w\}$ (where u, v are the endpoints of e) has $\ell(w) = d-1$. Each of u, v has $\ell \in \{d, d+1\}$ (from $V(f) \subseteq L_d \cup L_{d+1}$) and is adjacent to w, forcing $\ell(u), \ell(v) \in \{d-2, d-1, d\} \cap \{d, d+1\} = \{d\}$.
- If $\delta_G(d_{f'}) = d+1$, then all three vertices of f' lie in $L_{\geq d+1}$, so in particular $\ell(u) = \ell(v) = d+1$.

Each connected boundary component thus carries a single type at every edge: any vertex on a boundary component has two boundary edges incident to it (by R1, see below), both of the same type, so its level is fixed. Therefore each boundary component of $\partial R_{C'}$ is monochromatic in level.

Boundary structure. Each connected component of $\partial R_{C'}$ traces a closed walk in G that, by the monochromaticity above, lies entirely in L_d or entirely in L_{d+1}. By Proposition 1.9, the depth-d faces of $F_{C'}$ at any $v \in L_d \cap V_{C'}$ form a single contiguous arc in v's rotation, so the source-side boundary walk visits each L_d-vertex of $V_{C'}$ exactly once: it is a simple cycle. At vertices $v \in L_{d+1} \cap V_{C'}$ the depth-d faces may split into multiple arcs of v's rotation; this corresponds exactly to v being a cut-vertex of O, and the inner-side boundary walk visits v correspondingly many times — which is already accommodated by Definition 1.5 (where B_{in} is the outer-face boundary closed walk of O, not necessarily a simple cycle).

Outer boundary. Because S lies on the outer face of Π_G, the boundary curve(s) of $R_{C'}$ on the L_d side are closer to S in the embedding. In the inherited embedding of C, the unique unbounded face is the merged region containing the rest of Π_G outside $R_{C'}$ on the S side, so its boundary — a simple cycle on L_d (or a single vertex when $V_{C'} \cap L_d = \{v_0\}$, the $d = 0$ case) — serves as B_{out}. We set $B_{\text{out}} := G[V_{C'} \cap L_d]$ if this is a cycle, and the single vertex $\{v_0\}$ in the degenerate case.

Inner outerplanar graph. By Lemma 2.6 of [1], $G[V_{C'} \cap L_{d+1}]$ is outerplanar. We set $O := G[V_{C'} \cap L_{d+1}]$. The boundary curve(s) of $R_{C'}$ on the L_{d+1} side are exactly the boundary of O's outer face in the inherited embedding; this outer-face boundary is a single closed walk that traces around O from the outside, traversing any bridge edge twice and visiting cut-vertices multiple times. This walk is the inner boundary B_{in}. No further restriction on O's internal structure is needed: when $R_{C'}$ has more than two boundary components in the surface-classification sense (i.e. several disjoint simple cycles on L_{d+1}), these correspond precisely to the multiple connected components or bridge crossings of O, and the outer-face boundary closed walk of O captures them collectively.

Tire structure. The triangular faces of C inside the closed boundary region are by construction the depth-d faces in $F_{C'}$, and the edges of C are $E(B_{\text{out}}) \cup E(O) \cup E_{\text{ann}}$

where E_{ann} are the edges of G between $V_{C'} \cap L_d$ and $V_{C'} \cap L_{d+1}$ that bound a face of $F_{C'}$. □

Remark 1.11. Either boundary part of C in Lemma 1.10 may be degenerate. At $d = 0$ with single-vertex source $S = \{v_0\}$ the unique component of G'_0 has $V_{C'} \cap L_0 = \{v_0\}$ as the degenerate *outer* boundary and $V_{C'} \cap L_1$ a cycle (the link of v_0 in G) as the inner boundary. Symmetrically, at $d = D_{\max}$, $V_{C'} \cap L_{D_{\max}+1} = \emptyset$ degenerates to a single deepest vertex serving as the *inner* boundary, with the level-$D_{\max}$ cycle as the outer boundary.

Remark 1.12. Two structural features of $R_{C'}$ that might at first appear to obstruct the tire-graph conclusion are both already accommodated by Definition 1.5:

Cut-vertices of O. A vertex $v \in V_{C'} \cap L_{d+1}$ may have the faces of $F_{C'}$ incident to it split into two or more arcs in v's rotation in Π_G, separated by faces of higher depth. This corresponds exactly to v being a cut-vertex of $O = G[V_{C'} \cap L_{d+1}]$, and the inner boundary closed walk B_{in} then visits v multiple times — once for each arc. No additional hypothesis is needed.

Multi-hole topology of $R_{C'}$. Even when $R_{C'}$ encloses several disjoint depth-$> d$ sub-regions, the inner outerplanar graph O captures the multi-hole structure as a disconnected or non-2-connected outerplanar graph (with bridges or multiple components), and its outer-face boundary closed walk serves as B_{in} traversing bridges twice and visiting cut-vertices multiple times.

In the special case $d = 0$ with single-vertex source $S = \{v_0\}$, $R_{C'}$ is the star of v_0, a topological closed disk with one boundary cycle (the link of v_0); the corresponding tire graph has degenerate outer boundary $\{v_0\}$.

Proposition 1.13 (Edge–vertex coloring bijection for $D(T)$)**.** *Let T be a tire graph satisfying the spoke-only hypothesis of Proposition 1.8 (so $D(T) \cong C_{n+m} \circ K_1$). Let $\Gamma \cong C_{n+m}$ be the interior dual subgraph of $D(T)$ induced on the interior dual vertices $\{d_f : f \in F_{\mathrm{ann}}\}$. Then the number of proper 3-edge-colorings of $D(T)$ equals the number of proper 3-vertex-colorings of Γ, both given by*

$$2^{n+m} + 2 \cdot (-1)^{n+m}.$$

Proof. Write $L = n + m$, $\Gamma = C_L$. We construct mutually inverse bijections.

Step 1: proper 3-edge-colorings of $D(T) \leftrightarrow$ proper 3-edge-colorings of C_L. Given a proper 3-edge-coloring χ of $D(T)$, the three edges incident to any d_f carry three distinct colors; in particular the two cycle edges incident to d_f carry distinct colors, so $\chi|_{E(C_L)}$ is a proper 3-edge-coloring of C_L. Conversely, given a proper 3-edge-coloring ψ of C_L, the two cycle edges at any d_f have distinct colors, so a unique third color is available; assign that color to d_f's leaf edge. The resulting extension to $D(T)$ is proper at every d_f and vacuously proper at every leaf (degree 1), and the two maps are inverse to each other. Therefore

$$\#\bigl\{\text{proper 3-edge-colorings of } D(T)\bigr\} \;=\; \#\bigl\{\text{proper 3-edge-colorings of } C_L\bigr\}.$$

Step 2: proper 3-edge-colorings of $C_L \leftrightarrow$ proper 3-vertex-colorings of $L(C_L) \cong C_L$. The line graph $L(C_L)$ of a cycle of length L is again a cycle of length L; proper edge-colorings of C_L are by definition proper vertex-colorings of $L(C_L)$.

Step 3: count. The chromatic polynomial of the cycle is $P(C_L, k) = (k-1)^L + (-1)^L(k-1)$; at $k = 3$ this gives $2^L + 2 \cdot (-1)^L$. □

Remark 1.14. Proposition 1.13 reduces counting proper 3-edge-colorings of $D(T)$ to counting proper 3-vertex-colorings of a single cycle, giving a closed form $2^{n+m} + 2(-1)^{n+m}$ that depends only on $n + m$ (not on the specific spoke-only annular triangulation, nor on the chord structure of O). The count is preserved under the corona-with-K_1 structure of Proposition 1.8 precisely because each degree-1 leaf imposes no proper-edge-coloring constraint on itself; its color is freely determined as the missing third color at its attached interior vertex.

References

[1] E. Bauerfeld, *Plane Depth Sequencing*, manuscript (math-research repository), 2026.