What "2-SAT solvability" means in the rainbow proof

2-SAT in general

2-SAT is short for *2-satisfiability*. Standard logical problem:

- You have a set of *boolean variables* $x_1, x_2, \ldots, x_n$, each taking value 0 or 1 (false/true).
- You have a set of *clauses*, each a constraint involving exactly *two* variables.
- **2-SAT solvability** = "does there exist an assignment of 0/1 to all variables such that every clause is satisfied simultaneously?"

2-SAT is polynomial-time decidable (linear time, in fact) using the *implication graph* and *strongly-connected components*. This is in contrast to 3-SAT (3 variables per clause), which is NP-complete.

In our setup (the rainbow proof)

Variables. $o_0, o_1, \ldots, o_{m-1} \in \{0, 1\}$, one *orientation bit* per D-position on the dual annular cycle T'_{ann}. At each D-position p_j, the two incident cycle edges must take the two colors of $\{1, 2, 3\} \setminus \{\sigma_j\}$ in some order; o_j decides which color sits on the left of p_j versus the right.

Clauses. One per inter-D-position gap on T'_{ann}. Each gap involves two adjacent orientations (o_j, o_{j+1}) and forbids certain combinations:

gap type	clause type	combos forbidden
length-1, $\sigma_j \neq \sigma_{j+1}$	equality (fixed pair)	3 of 4 forbidden, leaves one valid combo
length-1, $\sigma_j = \sigma_{j+1}$	$o_j = o_{j+1}$	$(0, 1)$ and $(1, 0)$ forbidden
length-2, $\sigma_j \neq \sigma_{j+1}$	one combo forbidden	1 of 4 forbidden
length-2, $\sigma_j = \sigma_{j+1}$	$o_j = o_{j+1}$	$(0, 1)$ and $(1, 0)$ forbidden
length-≥ 3	no clause	gap has full slack

Each clause involves exactly two variables, so the system is 2-SAT.

Cyclic structure. Because the gaps wrap cyclically around T'_{ann}, the clauses form a *cyclic* chain — clause on (o_0, o_1), then (o_1, o_2), ..., finally (o_{m-1}, o_0).

What "solvable" means concretely

For a given $\sigma \in \mathcal{P}_m$:

- **Solvable** $\Leftrightarrow$ there is an orientation $(o_0, \ldots, o_{m-1})$ satisfying all clauses simultaneously $\Rightarrow$ proper edge 3-coloring of $T'_{f'}$ inducing σ exists $\Rightarrow$ $\sigma \in \pi_D$.

- **Unsolvable** $\Leftrightarrow$ no such orientation $\Rightarrow \sigma \notin \pi_D$.

So the rainbow proof's **Conjecture 1.5 (2-SAT solvability)** says: for every σ with both halves a permutation of $\{1,2,3\}$, the resulting cyclic 2-SAT system has at least one satisfying assignment.

Why a cyclic 2-SAT chain can fail

A 2-SAT cycle *can* be unsolvable when the implications chain around and force a contradiction. Toy example with 3 variables on a cycle:

- Clause 1: $o_0 \neq o_1$.
- Clause 2: $o_1 \neq o_2$.
- Clause 3: $o_2 \neq o_0$.

Implications:

$$o_0 = 0 \;\Rightarrow\; o_1 = 1 \;\Rightarrow\; o_2 = 0 \;\Rightarrow\; o_0 = 1.$$

Contradiction. So this 3-cycle of "not-equal" clauses is unsatisfiable (it is the chromatic-number obstruction: a triangle has no proper 2-coloring).

Why we believe our cyclic chain doesn't fail

For our m-variable cyclic 2-SAT (with clauses determined by $\sigma \in \mathcal{P}_m$), the analogous question is: does the specific pattern of clauses ever create such a wrap-around contradiction?

Empirically: no. For $m = 6$, $m_1 \in \{5,6,7,8\}$ and all 36 elements of $\mathcal{P}_6$, the cyclic 2-SAT system always has between 6 and 18 satisfying assignments (computed in `experiments/orbit_decomposition.py`).

Structurally: the constraints have a special form (forbidden combos depend on σ_j, σ_{j+1} via the third color $u = \{1,2,3\} \setminus \{\sigma_j, \sigma_{j+1}\}$). The hope is that a parity or implication-graph argument shows the cyclic chain cannot consistently wrap around to a contradiction when σ has the perms-per-half shape. This argument is what the proof of Conjecture 1.5 would supply, and what is currently missing.

Why we care

Proving Conjecture 1.5 would convert `rainbow_proof.tex`'s "provisional corollary" into a genuine theorem:

$$\pi_D(\mathcal{C}(T)) = \mathcal{P}_m \quad \text{whenever } m_1 \geq m-1,$$

which in turn upgrades the chain-pigeonhole step at $|\gamma| = m$ to a structural claim of the form "$\pi_U(\mathcal{C}(T_2))$ meets the fixed 36-element set $\mathcal{P}_m$," a strictly smaller and more tractable overlap condition than "π_U meets π_D."