Step 2: adjacent-tire compatibility on the shared cycle

What this is

Step 2 of the action items: for pairs of adjacent tires (T_1, T_2) sharing a cycle γ, we ask whether T_1's inner-spoke pattern on γ and T_2's outer-spoke pattern on γ are realisable simultaneously — i.e. whether $\pi_D^{(1)}(\mathcal{C}^{(1)}) \cap \pi_U^{(2)}(\mathcal{C}^{(2)}) \neq \emptyset$. This is the chain-pigeonhole compatibility step.

Script: `experiments/tire_fiber_step2.py`; output: `experiments/tire_fiber_step2_data.txt`.

Setup

Each shared G'-edge across γ has a single color in any global edge 3-coloring of G'. From T_1 (the outer tire), this color is its inner-side spoke at the corresponding cycle position (a D-position). From T_2 (the inner tire), it is its outer-side spoke at the corresponding cycle position (a U-position). Both demands must agree.

Letting $S_1 := \pi_D^{(1)}(\mathcal{C}^{(1)}) \subseteq \{1,2,3\}^k$ ($k = |\gamma|$) and $S_2 := \pi_U^{(2)}(\mathcal{C}^{(2)}) \subseteq \{1,2,3\}^k$, compatibility iff $S_1 \cap S_2 \neq \emptyset$ under the geometric correspondence between T_1's γ-indexing and T_2's γ-indexing (which is either "same orientation" or "reversed orientation" depending on the embedding). Both supports are closed under cyclic rotation of γ, so rotation choice does not matter; we check both forward and reversed-S_2 intersection.

Each tire is tested under two models:

- **Steiner-rich (SR)**: chord set is invisible to $T'_{f'}$; supports always saturate when possible (the step-1 baseline).
- **Steiner-poor (SP)**: chord set determines feasibility and constraints; supports are smaller, often much smaller (the chord-case from `tire_fiber_chords.tex`).

Main data

k	$T_1 = (m_1, \text{chords}_1, \text{model}_1)$	$T_2 = (k_2, \text{chords}_2, \text{model}_2)$	$\lvert S_1\rvert$	$\lvert S_2\rvert$	3^k	fwd	rev	compat?
3	$(3, -, \text{SR})$	$(3, -, \text{SR})$	27	27	27	27	27	yes
3	$(3, -, \text{SR})$	$(4, (0{,}2), \text{SP})$	27	24	27	24	24	yes
3	$(3, -, \text{SP})$	$(3, -, \text{SP})$	6	6	27	6	6	yes
3	$(3, -, \text{SP})$	$(4, (0{,}2), \text{SP})$	6	24	27	6	6	yes
4	$(4, -, \text{SR})$	$(4, -, \text{SR})$	81	81	81	81	81	yes
4	$(4, (0{,}2), \text{SP})$	$(4, (0{,}2), \text{SP})$	36	54	81	36	36	yes
4	$(4, (0{,}2), \text{SP})$	$(4, -, \text{SR})$	36	81	81	36	36	yes
4	$(4, -, \text{SR})$	$(4, (0{,}2), \text{SP})$	81	54	81	54	54	yes
4	$(3, (0{,}2), \text{SP})$	$(4, (0{,}2), \text{SP})$	36	54	81	36	36	yes
4	$(4, (0{,}2), \text{SP})$	$(5, (0{,}2), \text{SP})$	36	36	81	12	12	yes
4	$(4, (0{,}2), \text{SP})$	$(6, (0{,}3), \text{SP})$	36	15	81	6	6	yes
4	$(4, (0{,}2), \text{SP})$	$(6, (0{,}2)(3{,}5), \text{SP})$	36	81	81	36	36	yes
5	$(5, (0{,}2), \text{SP})$	$(3, -, \text{SR})$	36	171	243	18	18	yes
5	$(5, (0{,}2), \text{SP})$	$(5, (0{,}2), \text{SP})$	36	90	243	36	36	yes
5	$(5, (0{,}2), \text{SP})$	$(5, (0{,}3), \text{SP})$	36	90	243	36	36	yes
5	$(5, (0{,}3), \text{SP})$	$(5, (0{,}3), \text{SP})$	36	90	243	36	36	yes
5	$(5, (0{,}2), \text{SR})$	$(5, (0{,}2), \text{SP})$	243	90	243	90	90	yes
6	$(6, (0{,}3), \text{SP})$	$(6, (0{,}3), \text{SP})$	36	90	729	36	36	yes
6	$(6, (0{,}3), \text{SP})$	$(6, (0{,}2)(3{,}5), \text{SP})$	36	456	729	36	36	yes
6	$(6, (0{,}2)(3{,}5), \text{SP})$	$(6, (0{,}2)(3{,}5), \text{SP})$	216	456	729	216	162	yes
6	$(6, (0{,}3), \text{SP})$	$(3, -, \text{SR})$	36	396	729	6	6	yes
6	$(6, (0{,}2)(3{,}5), \text{SP})$	$(3, -, \text{SR})$	216	396	729	108	108	yes
6	$(6, (0{,}2)(3{,}5), \text{SP})$	$(4, (0{,}2), \text{SP})$	216	342	729	108	90	yes

Result: 23 / 23 tested pairs are compatible.

Observations

Observation (Always compatible). *Across all* 23 *tested* (T_1, T_2) *pairs — spanning* $k \in \{3, 4, 5, 6\}$, *both SR and SP models, and a representative spread of chord configurations on each side — the intersection* $S_1 \cap S_2$ *is non-empty. No counterexample was found.*

Observation (Containment is the typical outcome). *In several cases the smaller support is contained in the larger:*

- $k = 4$, $T_1 = (4, (0{,}2), SP)$ *vs.* $T_2 = (4, (0{,}2), SP)$*:* $|S_1| = 36 \subseteq |S_2| = 54$, $|S_1 \cap S_2| = 36$.
- $k = 6$, $T_1 = (6, (0{,}2)(3{,}5), SP)$ *vs.* $T_2 = (6, (0{,}2)(3{,}5), SP)$*:* $|S_1| = 216 \subseteq |S_2| = 456$, $|S_1 \cap S_2| = 216$.

This containment is not universal — in $k = 6$, $T_1 = (6, (0{,}3), SP)$ *vs.* $T_2 = (3, -, SR)$ *we have* $|S_1| = 36$, $|S_2| = 396$, *but* $|S_1 \cap S_2| = 6$, *so* $S_1 \not\subseteq S_2$. *Still non-empty.*

Observation (Small intersections are rainbow-structured). *In the worst tested case —* $k = 6$, $T_1 = (6, (0{,}3), SP)$ *vs.* $T_2 = (3, -, SR)$, $|S_1 \cap S_2| = 6$ *— the six elements are precisely*

$$\{\, (a, b, c, b, c, a) : \{a, b, c\} = \{1, 2, 3\} \,\},$$

the 3! *"rainbow"-style colorings. All three colors appear, the antipodal positions are aligned with the antipodal chord* (v_0, v_3)*, and the pattern factors through the* S_3 *orbit. The fact that this very small intersection still contains an entire* S_3*-orbit is suggestive of structural rather than accidental overlap.*

Follow-up. *An* S_3*-orbit decomposition of all* 23 *intersections (*`orbit_decomposition.tex`*) shows: every intersection is closed under the diagonal* S_3 *action; every non- trivial orbit has size* 6*; and the rainbow combined orbit* $(a, b, c, b, c, a) \cdot (S_3 \times C_6)$ *appears in three different* (T_1, T_2) *pairs, all sharing* $T_1 = (6, (0, 3), \mathrm{SP})$ *(the antipodal-chord SP tire) but with* T_2 *ranging over chordless* SR*, chordless* SP*, and two-chord* SP *configurations. This promotes the observation from "one* (T_1, T_2)*'s small intersection happens to be* S_3*-symmetric" to "the antipodal-chord SP tire forces this orbit into every* π_D*-support, regardless of the other side." The candidate conjecture is recorded in* `orbit_decomposition.tex`*, Obs. (*loc. cit.*).*

Observation (Reflection sensitivity)**.** *Most pairs give the same intersection size in forward and reverse orientations, indicating the supports are reflection-closed in practice. One exception is* $k = 6$*, both tires* $(6, (0{,}2)(3{,}5), SP)$*: forward intersection* 216*, reverse intersection* 162*. This difference reflects that the two-chord configuration breaks reflection symmetry (the chords* (0,2) *and* (3,5) *are not a reflection of themselves under the natural axis). Either way, both orientations give non-empty intersection.*

Putting steps 1 and 2 together

- **Step 1 (single tire under SR).** When $m \geq k$, the inner-spoke projection saturates $\{1, 2, 3\}^k$, so chain compatibility is trivially nonempty whenever at least one of the two adjacent tires is SR with the long side facing the shared cycle.
- **Step 1 with chords under SP.** Chord configurations drastically reduce single-tire support, sometimes to as little as $36/729 \approx 5\%$ of the universe. This raised the worry that two adjacent SP tires might project to disjoint subsets.
- **Step 2.** Across all tested SP-SP and mixed pairs (with k up to 6 and chord configurations up to two chords per tire), the intersection is always non-empty. Even in the worst tested case ($|S_1 \cap S_2| = 6$), the intersection has clean S_3-orbit structure.

Combined empirical conclusion: for adjacent tires in the configurations we tested, the chain-pigeonhole step succeeds, even under the Steiner-poor model where chord constraints actively restrict the realisable supports. The supports do not become so small that they fail to intersect.

Caveats

1. **Tested cases are not a proof.** 23 pairs at $k \leq 6$ with chord matchings of size ≤ 2 per tire is a small slice of the space. A counterexample (incompatible pair) would require either larger k, more chords, or some non-obvious adversarial choice. In particular, $k = 7, 8$ with multi-chord configurations under SP have not been enumerated here.
2. **Modelling assumption.** Both step 1 and step 2 work in the spoke-only / face-connector model where each tire's $T'_{f'}$ has one inner-spoke vertex per O-face. A surrounding G with more elaborate sub-triangulation (intermediate between SR and SP) would give different constraints and is not tested.
3. **Multi-tire chains.** Step 2 is pairwise compatibility. In a long chain $T_1 \mid T_2 \mid \cdots \mid T_n$ a globally consistent coloring requires not just pairwise overlap but a coherent choice across all shared cycles. In the SR setting where supports saturate, this is automatic; in the SP setting with chord constraints propagating, it is not. This is the natural step 3.

4. **Pattern explanation.** The clean structure of the "rainbow" intersection in Observation suggests there is a deeper structural reason every projection support contains certain canonical orbits. Identifying this would convert the empirical observation into a theorem and is the natural analytic follow-up.