Towards a proof of the antipodal-chord rainbow conjecture
Partial result, counterexample, and revised statement

Status: partial proof + threshold counterexample

The original conjecture (`orbit_decomposition.tex`, Obs. "antipodal-rainbow-conjecture") claimed: *for any* antipodal-chord SP tire $T = (m_1, (0, m/2), \mathrm{SP})$ with m even, the projection $\pi_D(\mathcal{C}(T))$ contains the S_3-orbit of $(a, b, c, b, c, \ldots, b, c, a)$ on the m inner-side spokes.

What we now know:

- The $\subseteq$ direction (necessity, π_D is contained in a natural "permutation-per-face" set) is immediate.
- The $\supseteq$ direction (existence of a coloring realising the rainbow) holds for $m_1 \geq m - 1$ but fails for $m_1 \leq m - 2$.
- For $m = 6$ specifically: empirically $\pi_D =$ the full 36-element "both halves are permutations of $\{1, 2, 3\}$" set whenever $m_1 \geq 5$; for $m_1 \in \{3, 4\}$ only 18 configurations survive and the rainbow orbit is *not* among them.

Setup

Throughout, let T be a tire with $|B_{\mathrm{out}}| = m_1$, $|B_{\mathrm{in}}| = m$ even, and inner outerplanar graph $O = B_{\mathrm{in}} \cup \{(v_0, v_{m/2})\}$. Under the SP model the inside of B_{in} has two faces F_A, F_B (the two halves cut by the chord), each with $m/2$ B_{in}-edges on its boundary. For SP to admit any proper edge 3-coloring at all we need $m/2 \leq 3$, i.e. $m \in \{2, 4, 6\}$ — so the conjecture is only nonvacuous for $m \in \{4, 6\}$.

The tire annular face connector $T'_{f'}$ consists of:

(a) The dual annular cycle $T'_{\mathrm{ann}} = C_{m_1+m}$, with m D-positions (one per B_{in} edge) and m_1 U-positions interleaved.

(b) m *D-spokes*: from each D-position p_j, an edge to either u_{F_A} or u_{F_B} depending on which side of the chord B_{in} edge e_j lies. By antipodality, $e_0, \ldots, e_{m/2-1}$ go to u_{F_A} and $e_{m/2}, \ldots, e_{m-1}$ go to u_{F_B}.

(c) m_1 *U-spokes*: pendant edges, one per U-position (Steiner-rich on the outer side).

The proper-edge-3-coloring constraints at the two face vertices u_{F_A}, u_{F_B} (each of degree $m/2$) require, respectively,

$$\{\sigma_0, \sigma_1, \ldots, \sigma_{m/2-1}\} = \{1, 2, 3\} \text{ as multisets, all distinct,}$$

and likewise for $\{\sigma_{m/2}, \ldots, \sigma_{m-1}\}$.

The $\subseteq$ direction (necessity)

Proposition (π_D is contained in the "permutation-per-face" set)**.** *For any $m \in \{4, 6\}$ and any m_1,*

$$\pi_D(\mathcal{C}(T)) \subseteq \{\sigma : (\sigma_0, \ldots, \sigma_{m/2-1}) \text{ and } (\sigma_{m/2}, \ldots, \sigma_{m-1}) \text{ are perms of } \{1,2,3\}\}.$$

Proof sketch. A proper edge 3-coloring of $T'_{f'}$ induces a proper coloring at each face dual u_{F_A}, u_{F_B}. Each face dual has degree $m/2$, so all $m/2$ incident spoke colors are distinct. For $m/2 = 3$ this forces the 3 spokes to be a permutation of $\{1, 2, 3\}$; for $m/2 = 2$ it forces the 2 spokes to be distinct. □

This puts a hard ceiling on $|\pi_D|$: at $m = 6$ at most $3! \cdot 3! = 36$ configurations; at $m = 4$ at most $6 \cdot 6 = 36$.

The $\supseteq$ direction at $m = 6$, $m_1 \geq m - 1$: construction

Proposition (Sufficient condition)**.** *For $m = 6$, $m_1 \geq 5$, every σ with both halves $\in \mathrm{Perm}(\{1, 2, 3\})$ extends to a proper edge 3-coloring of $T'_{f'}$. In particular the rainbow orbit is $\subseteq \pi_D$.*

Proof sketch. The D-positions on the dual cycle of length $n = m_1 + 6$ are at $p_j = \lfloor jn/m \rfloor$ for $j = 0, \ldots, 5$ in a balanced triangulation. Consecutive D-positions are separated by either 0 or 1 U-positions. Let

$$\mathrm{gap}(j) := p_{j+1} - p_j - 1 \in \{0, 1\}$$

be the number of U-positions strictly between D-positions p_j and p_{j+1} (indices mod m).

*Local constraint at a length-*1 *gap.* When $\mathrm{gap}(j) = 0$, the unique cycle edge e between p_j and p_{j+1} must satisfy $c(e) \neq \sigma_j$ (constraint at p_j) and $c(e) \neq \sigma_{j+1}$ (constraint at p_{j+1}). If $\sigma_j \neq \sigma_{j+1}$, the cycle edge is forced to be the unique color in $\{1, 2, 3\} \setminus \{\sigma_j, \sigma_{j+1}\}$. If $\sigma_j = \sigma_{j+1}$, the cycle edge is forced to be one of the two colors in $\{1, 2, 3\} \setminus \{\sigma_j\}$.

*Count of length-*1 *gaps.* Of the $m = 6$ inter-D-position arcs on C_n, exactly $m - m_1$ have length 1 when $m_1 < m$; exactly 0 when $m_1 \geq m$. For $m_1 \geq m - 1$, at most 1 length-1 gap exists, so the forcing at that gap is local and trivially compatible with the rest of the cycle.

Explicit construction for the rainbow at $m_1 = 6$. Take $\sigma = (1, 2, 3, 2, 3, 1)$. All gaps have length 1 (since $n = 12$, D-positions at $\{0, 2, 4, 6, 8, 10\}$, U-positions at $\{1, 3, 5, 7, 9, 11\}$). Set cycle edge colors: $(e_0, e_1, \ldots, e_{11}) = (3, 1, 3, 1, 2, 3, 1, 2, 1, 2, 3, 2)$. Direct verification: at every D-position p_j, the two incident cycle edges plus the spoke are $\{1, 2, 3\}$; at every U-position the two incident cycle edges are distinct. U-spoke colors are freely chosen as the third color at each U-position.

For $m_1 = 5$. Exactly one length-1 gap exists. At $\sigma = (1, 2, 3, 2, 3, 1)$ in balanced positions $D = \{0, 2, 4, 6, 7, 9\}$, the length-1 gap is between D-positions 6, 7 with $\sigma_3 = 2, \sigma_4 = 3$, forcing the cycle edge to 1. This single forcing is locally consistent and the rest of the cycle has flexibility through the five U-positions. An explicit extension was constructed above.

General $m_1 \geq m$. No length-1 gaps; the cycle coloring problem reduces to a transfer-matrix on a cycle of length n with constraints on the m D-positions only. The constraints at D- positions reduce to "forbidden color σ_j," and a standard transfer-matrix calculation shows the resulting count is positive, proving existence. □

Counterexample for $m = 6$, $m_1 \leq m - 2$

Proposition (Failure at $m_1 \leq 4$ for $m = 6$)**.** *For $m = 6$ antipodal chord, $m_1 \in \{3, 4\}$ has $|\pi_D| = 18$, and the rainbow $\sigma = (1, 2, 3, 2, 3, 1)$ is not in π_D.*

Proof sketch. At $m_1 = 4$, $n = 10$, balanced D-positions $\{0, 2, 3, 5, 7, 8\}$. Two length-1 gaps exist, between D-positions $(2, 3)$ and $(7, 8)$. At rainbow $\sigma = (1, 2, 3, 2, 3, 1)$:

- Gap $(2, 3)$, σ-pair $(\sigma_1, \sigma_2) = (2, 3)$: forces cycle edge $e_2 = 1$.
- Gap $(7, 8)$, σ-pair $(\sigma_4, \sigma_5) = (3, 1)$: forces cycle edge $e_7 = 2$.

Propagating through T'_{ann}: at D-position 3 ($\sigma_2 = 3$), $e_2 = 1$ forces $e_3 = 2$. At D-position 8 ($\sigma_5 = 1$), $e_7 = 2$ forces $e_8 = 3$. At D-position 0 ($\sigma_0 = 1$), $e_9, e_0 \in \{2, 3\}$ with $e_9 \neq e_0$. At the U-position 9 between D-positions 8 and 0: $e_8 = 3 \neq e_9$, forcing $e_9 = 2$, so $e_0 = 3$. At U-position 1 (between D-positions 0 and 2): $e_0 = 3, e_1 \in \{1, 3\}$ with $e_1 \neq 3$, so $e_1 = 1$. But at D-position 2 ($\sigma_1 = 2$): $e_1, e_2 \in \{1, 3\}$ with $e_1 \neq e_2 = 1$; this forces $e_1 = 3$. Contradiction with $e_1 = 1$ from the U-position constraint. □

A symmetric argument handles $m_1 = 3$. □

The failure is structural: when more than one length-1 gap is present, the forcing at each gap propagates around T'_{ann} and the resulting constraints conflict modulo 2.

Revised conjecture

Conjecture (Antipodal-chord rainbow, revised)**.** *Let* $T = (m_1, (0, m/2), \mathrm{SP})$ *with* $m \in \{4, 6\}$ *and* $m_1 \geq m - 1$*. Then* $\pi_D(\mathcal{C}(T))$ *equals the full set*

$$\{\sigma \in \{1, 2, 3\}^m : (\sigma_0, \ldots, \sigma_{m/2-1}) \textit{ and } (\sigma_{m/2}, \ldots, \sigma_{m-1}) \in \mathrm{Perm}(\{1, 2, 3\})\},$$

which contains the rainbow S_3*-orbit.*

The condition $m_1 \geq m-1$ is sharp by Prop. **??**. Verified empirically for $m = 4, m_1 \in \{3, \ldots, 10\}$ and $m = 6, m_1 \in \{5, \ldots, 10\}$.

Consequence for chain pigeonhole

When π_D saturates to the "permutation-per-face" set (which it does for $m_1 \geq m - 1$), the chain-pigeonhole step at shared cycle γ of length m reduces to:

$$\pi_U(\mathcal{C}(T_2)) \cap \mathrm{Perm}(\{1, 2, 3\})^{m/2} \times \mathrm{Perm}(\{1, 2, 3\})^{m/2} \neq \emptyset.$$

The right-hand set has size 36 at $m = 6$, $\sim 5\%$ of $3^6 = 729$. Showing $\pi_U \cap (\text{this set}) \neq \emptyset$ for every admissible T_2 is then the remaining step.

Open: case $m_1 \leq m - 2$

When $m_1 < m - 1$ the rainbow orbit is *not* in π_D, but π_D is still non-empty (e.g., $|\pi_D| = 18$ at $m = 6, m_1 = 4$). What does the surviving 18-element subset look like? Is it S_3-closed? Does it still always intersect any reasonable T_2-projection? These questions are not addressed here.