Must a minimum 4CT counterexample have a separating n-cycle with n even and $n \geq 6$?

The question

Statement. Let G be a hypothetical minimum 4-colour counterexample (a minimum planar triangulation requiring ≥ 5 colours). Must G contain a separating n-cycle with n *even* and $n \geq 6$?

Short answer. I do not know of a proof either way. The question is subtle: a parity computation (below) shows that cuts in the cubic dual G^* have a definite parity tied to the side sizes, but I do not see this parity forcing the existence of even-length separating cycles.

Parity computation in the cubic dual

Let G be a planar triangulation and G^* its cubic planar dual. G has V vertices, $3V-6$ edges, $2V-4$ faces; G^* has $2V-4$ vertices, $3V-6$ edges, V faces. In particular, $|V(G^*)| = 2V-4$ is *always even.*

Lemma (Cut size parity in cubic graphs)**.** *Let G^* be a cubic graph and let C be an edge cut separating $V(G^*)$ into sides S and $T = V(G^*) \setminus S$. Then*

$$|C| \equiv |S| \equiv |T| \pmod 2.$$

Proof. Counting degree on side S: $3|S| = 2e_S + |C|$, where e_S is the number of edges with both endpoints in S. Hence $|C| = 3|S| - 2e_S \equiv |S| \pmod 2$. Since $|S| + |T| = |V(G^*)|$ is even, $|S| \equiv |T| \pmod 2$. □

Translating to primal cycles. Via the $G \leftrightarrow G^*$ duality, an n-cycle in G corresponds to an n-edge cut in G^*. Lemma says: an *even* n-cycle in G ($n \geq 6$) corresponds to a G^* cut with both sides having an *even* number of vertices. An *odd* n-cycle in G ($n \geq 7$) corresponds to both sides having an *odd* number of vertices.

What Birkhoff gives us

For a minimum 4CT counterexample G:

- G is internally 6-connected (no separating 3-cycle, no separating 4-cycle, no separating 5-cycle with ≥ 2 vertices on each side).
- Equivalently in G^*: cyclic edge connectivity ≥ 6.

So all sufficiently small cuts in G^* are excluded. The smallest non-trivial cuts can be 6-edge ($n = 6$, even) or 7-edge ($n = 7$, odd), and Birkhoff alone permits both.

A potential "no" — can G^* have only odd cuts ≥ 7?

In principle, the minimum non-trivial cut could be 7 (and all non-trivial cuts could be odd-length). In this case the minimum primal separating cycle has length 7 (odd), and the question's answer is "no."

Is this realisable? We need a planar cubic graph G^* satisfying:

- cyclic edge connectivity ≥ 7;
- all non-trivial cyclic edge cuts have odd size.

Lemma gives a constraint: if a G^* has *any* cut at all of size ≥ 6, its parity is fixed by side size. For "all cuts odd" to hold, no even-sized cut would separate. In a cubic graph this corresponds to the side sizes being *odd*. Since $|V(G^*)|$ is even (for any planar triangulation dual), the sides $(|S|, |V(G^*)| - |S|)$ have matching parity — both even or both odd. "All cuts odd" means "all cyclic separations have odd side sizes," which is possible in principle.

However: I don't know of a planar cubic graph that has *only* odd cyclic cuts. The known internally 6-connected examples (icosahedron's dual = dodecahedron, etc.) all have many even cuts of size 6 as well as odd cuts.

What the maximal-planar constraint forces

Maximal planar (= triangulation) gives strong constraints: $E = 3V - 6$, $F = 2V - 4$, $\sum \deg = 6V - 12$. Average degree approaches 6 from below. Internally 6-connected forces min degree ≥ 5.

Forced 12 degree-5 vertices. If all degrees are in $\{5, 6\}$:

$$5n_5 + 6n_6 = 6V - 12, \quad n_5 + n_6 = V \implies n_5 = 12.$$

So exactly 12 degree-5 vertices, with the remaining $V - 12$ of degree 6. For $V > 12$, at least some degree-5 vertex has a degree-6 link neighbour.

Second-link length. For vertex v of degree d in a triangulation, the second link (cycle of vertices at distance exactly 2) has length

$$L_2(v) = d + \sum_{u \in \mathrm{link}(v)} (\deg u - 5).$$

(Each link vertex contributes one "shared" boundary vertex with each link-cycle neighbour, plus $\deg u - 5$ private boundary vertices inside u's fan.) For the icosahedron ($d = 5$, all link degrees 5): $L_2 = 5$.

Second-link length does *not* pin down min cyclic cut. The pentakis dodecahedron ($V = 32$): every degree-5 vertex has all 5 link vertices of degree 6, so $L_2 = 5 + 5 = 10$ around degree-5 vertices. Around degree-6 vertices, $L_2 = 6 + 0 \cdot 2 + 1 \cdot 4 = 10$. So no second-link separator has length 6.

But the pentakis dodecahedron's dual (Buckminsterfullerene graph) *does* have 6-edge cyclic cuts — they arise as separations *not* surrounding a single vertex. So second-link length is just one source of cyclic cuts; longer constructions can yield smaller cuts.

Conclusion of this section. The maximal-planar constraint forces some structural relations but does not pin down the minimum non-facial cyclic cut to any specific value. A minimum 4CT counterexample could in principle have any min cut size ≥ 6.

What I can conclude

- **Parity is determined by side size** (Lemma). Even n-cycle separators in G correspond to even-sided cuts in G^*.
- **Birkhoff doesn't rule out odd cuts.** Minimum non-trivial cyclic cut in G^* could in principle be of any size ≥ 6.
- **No known proof that even $n \geq 6$ separators must exist** in min 4CT counterexamples.
- **Empirically**, in all tested internally 6-connected planar triangulations (icosahedron, pentakis dodecahedron, Holton–McKay duals), even 6-cycle separators with both sides ≥ 2 vertices exist in abundance.

Conjecture

Conjecture. *Every internally 6-connected planar triangulation G with $|V(G)| \geq 12$ has a separating n-cycle with n even and $n \geq 6$.*

Equivalently: every planar cubic graph with cyclic edge connectivity ≥ 6 and $|V| \geq 20$ has a cyclic edge cut of size 6.

This conjecture seems plausible based on the second-link heuristic, but I don't have a proof. A planar cubic graph that violates it would be a structural curiosity worth a name — a "cyclically 7-edge-connected planar cubic graph" — and I do not know an example.

Relevance to the cut-tire framework. If the conjecture holds, our cut-tire framework's domain assumption (= cyclic edge connectivity exactly 6 in G^*) is automatically satisfied by every minimum 4CT counterexample. If it doesn't hold, we'd need to either prove that the counterexample is not of the violating type, or extend the framework to higher-size cuts.

Could the minimum non-trivial cyclic cut be 8?

Yes, in principle. Birkhoff gives ≥ 6; nothing in the condition pins the value to 6. A planar cubic G^* with non-facial cyclic edge connectivity *exactly* 8 would have:

- No non-facial cyclic cut of size 6 (= no separating 6-cycle in G with ≥ 2 vertices each side).
- No non-facial cyclic cut of size 7 (= no separating 7-cycle in G with ≥ 2 vertices each side).
- Some non-facial cyclic cut of size 8.

By the cut-parity lemma, a size-8 cut would have even-sized sides. Size-7 cuts would have odd-sized sides; for such cuts to not exist non-facially, the graph would need a structural parity barrier or just lack any odd-cardinality separations.

What would force this? Looking at the second-link heuristic: if every vertex's link contains only vertices of degree ≥ 6 rather than the icosahedron-tight degree 5, the second-link length jumps to $\geq 5 + 5 \cdot 1 = 10$. Such graphs exist (denser triangulations); whether such a graph is also a minimum 4CT counterexample (= class-2 cubic dual + planar + internally 6-connected) is unknown.

The framework adapts. Even if the minimum non-trivial cyclic cut is 8 (or some other value > 6), the cut-tire chain DP doesn't structurally depend on cut size $= 6$. The same constructions (cut tires, boundary cut tire T_∂, chain DP via shared edges) apply to 8-edge cuts with minor parameter changes. What *does* change: per-tire enumeration size scales with cut size, and the per-tire half (Prop 1.13) was proved specifically for spoke-only cut tires with simple-cycle face boundaries — it would need re-examining for larger structures.

Bottom line. The minimum non-trivial cyclic cut size for a hypothetical 4CT counterexample is one of $\{6, 7, 8, 9, \ldots\}$, and Birkhoff alone doesn't pin it down. The framework's natural domain is whichever value it happens to take, with 6 being the simplest case to enumerate and study.