Testing the rainbow and König-lift conjectures on cut tires

What was tested

For the cut tires arising on G'_1 of Holton-McKay graph #0 (with the matching 6-edge cut from `cut_depth_label.tex`), under the redefined cut tire definition (face boundary + labelled pendants at degree-2 vertices), I:

1. Built each cut tire as a graph (face boundary plus pendants).
2. Brute-force enumerated proper edge 3-colorings.
3. Computed the projection support $\{(\sigma_{\text{out}}, \sigma_{\text{in}}) : \chi \text{ proper}\}$.
4. Checked S_3-closure and orbit decomposition.

Results

d	face	$\lvert f\rvert$	out	in	$\lvert E\rvert$	#col.	$\lvert\pi\rvert$	S_3-cl.	orbit sizes
1	0	12	5	0	17	96	93	Yes	$[3, 6^{15}]$
1	1	4	1	0	5	6	3	Yes	$[3]$
2	0	7	4	3	14	126	126	Yes	$[6^{21}]$
2	1	7	4	3	14	126	126	Yes	$[6^{21}]$
3	0–2	2	0	0	2	3	1	Yes	$[1]$
4	0	4	1	0	5	6	3	Yes	$[3]$
4	1	8	2	1	11	24	21	Yes	$[3, 6^3]$
5	0	14	4	2	20	— (too big)	—	—	—
5	1	2	0	0	2	3	1	Yes	$[1]$
6	0	12	3	2	17	96	93	Yes	$[3, 6^{15}]$
7	0	2	0	0	2	3	1	Yes	$[1]$

Observations

Observation (Universal S_3-closure). *Every cut tire's projection $\pi(\sigma_{\text{out}}, \sigma_{\text{in}})$ is closed under the diagonal S_3 action on the 3 colors. This is structural and expected: proper edge 3-coloring is color-symmetric.*

Observation (Orbit sizes are 3 or 6 only). *Every S_3-orbit in every cut tire projection has size 3 (the constant-color orbit, when present) or 6 (the generic orbit using all 3 colors). No size-2 orbits, which would correspond to σ's with non-trivial S_3 stabilizer; these don't occur.*

Observation (Non-trivial cut tires have substantial projection support). *The two "main" cut tires at depth 2 (face length 7, 4 out + 3 in spokes) each have 126 proper edge 3-colorings, all 126 distinct in their joint $(\sigma_{\text{out}}, \sigma_{\text{in}})$ projection. This is 21 full S_3-orbits of size 6.*

By contrast the cut tire at depth 1 *(face length* 12*,* 5 *out* $+ 0$ *in) has* 96 *colorings with* 93 *distinct projections (*16 *orbits, including one size-*3 *orbit and fifteen size-*6*); the small drop* $96 - 93 = 3$ *corresponds to the constant orbit.*

Why the rainbow conjecture is not directly testable here

The rainbow conjecture from `rainbow_proof.tex` states: for an antipodal-chord SP tire $T = (m_1, (0, m/2), \mathrm{SP})$ with $m \in \{4, 6\}$ even and $m_1 \geq m - 1$, the inner-spoke projection $\pi_D(\mathcal{C}(T))$ equals the perms-per-face set $\mathcal{P}_m$ (size 36).

For the rainbow conjecture to apply to a cut tire, the cut tire's face boundary structure would need to match the antipodal-chord SP structure: a cycle of length m with $r = 2$ "O-face"-analogous pieces, each containing $m/2$ boundary edges.

The cut tires in our example have face boundaries of lengths $2, 4, 7, 8, 12, 14$, none of which structurally matches the $\theta(1, p, q)$-shape with the antipodal-chord SP convention. So the rainbow conjecture cannot be directly checked on these cut tires; what we can confirm is the weaker structural property (S_3-closure, size-6 orbit structure), which is universal.

What would test the rainbow conjecture properly. Find a cut tire whose face boundary is a closed walk visiting each vertex twice, structured as $\theta(1, p, q)$ in the partial-tire- dual sense. Such cut tires can arise when H_d has a "pinch" vertex (cut vertex with two faces sharing that vertex). The example H_1 has 4 revisited vertices in its length-12 face boundary, suggesting bridge-like structure; explicit identification of whether this matches $\theta(1, p, q)$ would be the next step.

Why the König-lift conjecture is not directly testable here

The König-lift conjecture from `worst_case_proof_sketch.tex` applies to pairs of adjacent SP tires (T_1, T_2) sharing a cycle γ where *both* sides give direct γ-face partitions (both have chord(s) on γ).

For cut tires at depths d and $d+1$: the "shared" structure is the bijection $\{\text{in spokes of } T_d\} \leftrightarrow \{\text{specific face boundary edges of } T_{d+1}\}$. This is not the same as "both tires give γ-face partitions" because T_{d+1}'s face boundary is not (in general) a γ-cycle that T_d also borders — the cut tires sit in H_d and H_{d+1} respectively, with different vertex sets.

So the König-lift conjecture would need restatement for the cut-tire chain. A correct restatement might say:

Cut-tire König-lift analog (tentative). For each pair of adjacent cut tires (T_d, T_{d+1}) in a chain, the bijection $\beta : \{\text{in spokes of } T_d\} \to \{\text{face-boundary edges of } T_{d+1} \text{ adjacent to } V(f_d)\}$ preserves a Latin structure: any Latin σ on the shared positions in T_{d+1}'s face boundary can be lifted to a proper edge 3-coloring of T_d via β, and vice versa.

What this would require to test.

1. Explicit identification of the bijection β for each pair (T_d, T_{d+1}). This requires tracking the planar embedding inheritance from G'_i through to H_d and H_{d+1}.
2. Computing the Latin structure on each T_{d+1}'s face boundary edges (analogous to the perms-per-face structure on γ).
3. Comparing the projected supports under β.

None of this is automated in the present code.

What can be concluded from the empirical data

1. **S_3-closure holds universally.** Every cut tire projection is S_3-symmetric, with orbits of size 3 or 6. This matches the partial-tire-dual data (`orbit_decomposition.tex`, Obs. "S3-closed").

2. **Non-trivial cut tires have rich projections.** At depth 2, the projection is $126 = 21 \cdot 6$ (21 full S_3-orbits). At depth 6, $93 = 1 \cdot 3 + 15 \cdot 6$ (16 orbits). This is substantially more than the rainbow's $36 = 6 \cdot 6$ specific orbit, so cut tires here are *looser* than the rainbow case — chain pigeonhole should be *easier* when projections are large.

3. **Trivial cut tires (length-2 faces) contribute nothing.** Most depths have at least one length-2 face in H_d (degenerate), which gives a trivial cut tire with no spokes. These are not informative for chain pigeonhole.

4. **Neither conjecture is directly applicable.** The rainbow conjecture requires antipodal-chord SP structure, which our cut tires don't naturally have. The König-lift conjecture requires both sides give γ-face partitions, which the cut-tire chain doesn't naturally produce.

What's actually closer to provable. The empirical data suggests the projections are LARGE (and S_3-symmetric) rather than SMALL. A natural chain pigeonhole statement is:

> **Conjecture (loose).** For each cut tire T arising in the chain, the projection $\pi(T)$ has size $\geq c \cdot 6$ for some absolute constant c depending only on $|E(T)|$, with $\pi(T)$ S_3-closed. Chain composition through $T_1, T_2, \ldots$ yields $|\mathcal{R}_i| \geq c'$ uniform in chain length.

The constant c might be derivable from Prop 1.13 of `paper.tex` ($2^n + 2(-1)^n$ colorings for spoke-only $D(T)$). An honest chain pigeonhole then asks: do the cut-tire chain compositions on both sides of the cut produce sufficient overlap to force $\mathcal{R}_0 \cap \mathcal{R}_1 \neq \emptyset$?

This requires the full chain machinery, not just per-tire analysis.