Birkhoff's condition: minimum 4CT counterexamples are internally 6-connected

The condition

Definition (Internally 6-connected)**.** *A planar triangulation G is* internally 6-connected *if every vertex cut of G of size ≤ 5 either fails to separate G, or is a 5-cut whose two sides have one side equal to a single vertex.*

Equivalently: G has no separating triangle (3-cycle), no separating quadrilateral (4-cycle), and any separating pentagon (5-cycle) isolates exactly one vertex on one side.

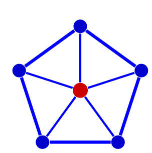
Theorem (Birkhoff 1913)**.** *If a minimum 4-colour counterexample (= minimum planar triangulation requiring more than 4 colours for a proper vertex colouring) exists, it must be internally 6-connected.*

The idea behind the proof: if G has a separating k-cycle for $k \in \{3, 4, 5\}$ with ≥ 2 vertices on each side, one can 4-colour each side independently and patch the colourings along the cycle, contradicting minimality. Only the 5-cycle isolating 1 vertex is irreducible by this argument.

Cases at a glance

Internally 6-connected = all four below:

• no separating 3-cycle	FORBIDDEN
• no separating 4-cycle	FORBIDDEN
• no separating 5-cycle isolating ≥ 2 on either side	FORBIDDEN
• separating 5-cycle isolating 1 vertex	ALLOWED

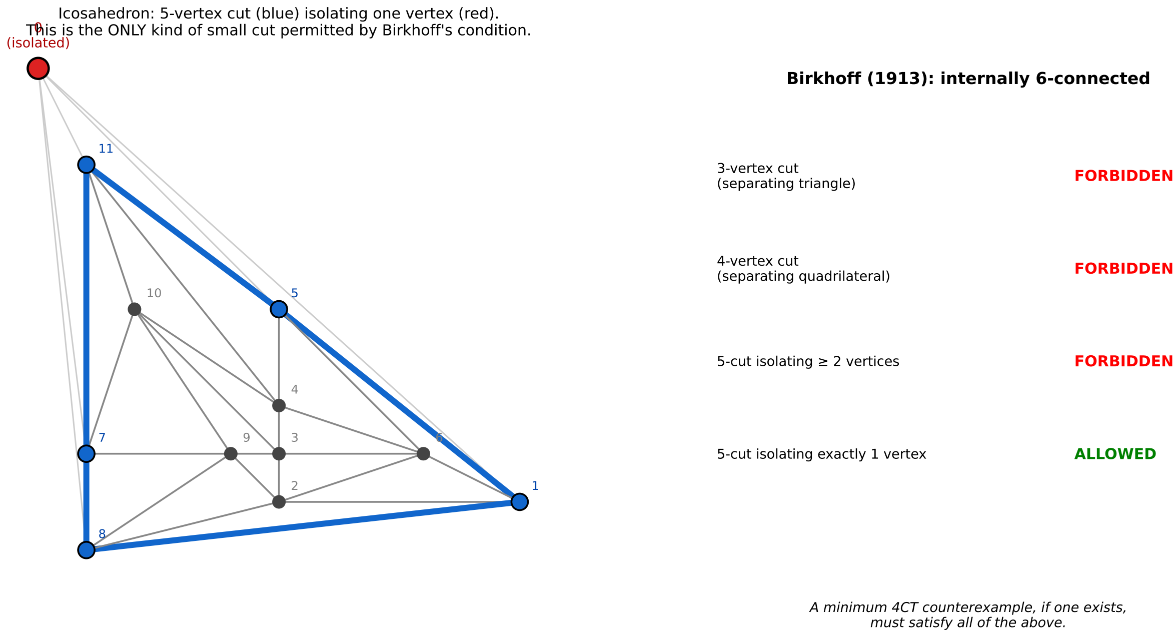


allowed case

The right-hand schematic shows the only permitted case: a vertex (red) surrounded by its 5 neighbours (blue) which form a 5-cycle. Removing the 5 blue vertices isolates the red one. This is what the next section's picture shows in the icosahedron.

Icosahedron: the smallest example

The icosahedral graph (the 1-skeleton of the regular icosahedron) has 12 vertices, all of degree 5, and is a planar triangulation. Every vertex's 5 neighbours form a 5-cycle (the "link" of the vertex); removing them isolates the vertex. This is the only kind of small cut: the icosahedron is internally 6-connected.



On the left, the icosahedron's planar (Schlegel) embedding: the red vertex (here vertex 0) is removed from the graph by the 5-cut consisting of its 5 blue neighbours $\{1, 5, 7, 8, 11\}$. The blue 5-cycle is the separator; on one side is the isolated red vertex, on the other are the remaining 6 vertices. This is the allowed Birkhoff configuration.

Internally 6-connected vs. cyclic edge connectivity

Birkhoff's condition rules out small *cuts* but does not force any specific separating *cycle* to exist. In the cubic dual G^*, the dictionary is:

- k-vertex cut in $G \leftrightarrow k$-edge cut in G^*.
- separating k-cycle in G (with ≥ 2 vertices on each side) $\leftrightarrow$ k-edge cut in G^* separating G^* into two pieces *each containing a cycle* (= *cyclic* edge cut).

So G internally 6-connected $\iff$ G^* has *cyclic edge connectivity* ≥ 6. Whether it is *exactly* 6 (= some separating 6-cycle in G exists with both sides ≥ 2 vertices) is an additional question.

Hidden assumption in the cut-tire framework. Our chain-DP framework (`chain_half_analysis.tex`, `boundary_cut_tire.tex`) operates on 6-edge cuts of the cubic dual. It therefore has nontrivial content on G only when G^* has cyclic edge connectivity *exactly* 6 — i.e. when G has a separating 6-cycle with ≥ 2 vertices on each side. Birkhoff gives "≥ 6"; we need "$= 6$".

This is a real *a priori* restriction: if a minimum 4CT counterexample turned out to have G^* with cyclic edge connectivity ≥ 7 (= no separating 6-cycle in G), our framework would be empty on it.

Empirically. All graphs in the test suite (icosahedron / dodecahedron, pentakis dodecahedron / Buckyball, Holton–McKay #0 through #5) have many 6-edge cuts in their cubic duals — i.e. all known internally 6-connected triangulations of moderate size have plenty of separating 6-cycles. Whether a hypothetical minimum 4CT counterexample *must* have one is a structural question discussed in the next note.

Why this matters for the framework

For our cut-tire chain DP framework, we test on graphs whose primal triangulation is internally 6-connected:

- **Icosahedron** ($V = 12$): smallest example. Its cubic dual = dodecahedron ($V = 20$).
- **Pentakis dodecahedron** ($V = 32$): icosahedron with each face subdivided. Cubic dual = truncated icosahedron = "Buckminsterfullerene" ($V = 60$).
- **Holton–McKay graphs** ($V = 21$ primal triangulation, $V = 38$ cubic dual): the smallest non-Hamiltonian internally-6-connected cubic plane graphs, candidates for a minimum counterexample in Tait's reduction even though all are 3-edge-colourable.

If our framework's claims hold for all internally 6-connected triangulations, then a minimum 4CT counterexample (if it existed) would be in the framework's domain — and any structural obstruction the framework finds would refute the counterexample's existence.

Sanity check

Sage verification (in `experiments/draw_internally_6_connected.py`):

```
Vertex 0 has 5 neighbors: [1, 5, 7, 8, 11]
Induced subgraph on neighbors: 5 edges, is_cycle=True
After removing the 5 neighbors: 2 components, sizes=[1, 6]
```

So vertex 0's neighbourhood is indeed a 5-cycle, removing it isolates vertex 0 from the other 6 vertices. By symmetry the same holds at every vertex of the icosahedron.