

DUAL DECOMPOSITION OF MINIMAL COUNTEREXAMPLES

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ABSTRACT.

1. THE MINIMAL COUNTEREXAMPLE

Throughout, a *triangulation* is a simple plane graph, with a fixed embedding, in which every face — including the outer face — is bounded by a triangle. We first reduce to triangulations, then record the degree properties a smallest counterexample must have.

Lemma 1.1 (Reduction to triangulations). *If every triangulation is properly 4-vertex-colourable, then so is every plane graph.*

Proof. Let H be a plane graph. Add edges to H , maintaining planarity, until no further edge can be added; the result is a triangulation H^+ on the same vertex set with $E(H) \subseteq E(H^+)$. A proper 4-colouring of H^+ restricts to a proper 4-colouring of H , since every edge of H is an edge of H^+ . \square

By Lemma 1.1, if the Four Colour Theorem fails then it fails for some triangulation. We may therefore make the following assumption.

Definition 1.2 (Minimal counterexample). Let G be a triangulation on the fewest vertices that admits no proper 4-vertex-colouring. We call G a *minimal counterexample*. By minimality, every triangulation on fewer than $|V(G)|$ vertices is properly 4-colourable.

Remark 1.3. Since every triangulation on at most four vertices is properly 4-colourable (the largest being K_4), a minimal counterexample has $|V(G)| \geq 5$; the degree bound below sharpens this to $|V(G)| \geq 12$.

Lemma 1.4 (Minimum degree). *A minimal counterexample G has minimum degree $\delta(G) \geq 5$.*

Proof. Suppose some vertex v has $\deg(v) = d \leq 4$.

If $d \leq 3$, let $G' = G - v$. Then G' is a plane graph on fewer vertices, so by Definition 1.2 and Lemma 1.1 it has a proper 4-colouring. The at most three neighbours of v use at most three colours, so a fourth colour is free for v , extending the colouring to G — a contradiction.

If $d = 4$, again 4-colour $G - v$. If the four neighbours of v use at most three colours we extend as before, so assume they receive all four colours; let v_1, v_2, v_3, v_4 be the neighbours in cyclic order around v , coloured 1, 2, 3, 4. Consider the subgraph

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induced by the colour classes 1 and 3, and let K be its connected component containing v_1 . If $v_3 \notin K$, swap colours 1 and 3 on K ; now no neighbour of v is coloured 1, freeing it for v . If $v_3 \in K$, then a 1–3 Kempe chain joins v_1 to v_3 , and this chain together with v encloses exactly one of v_2, v_4 ; hence the 2–4 component containing v_2 cannot also reach v_4 , and swapping colours 2 and 4 on it frees colour 2 for v . Either way the colouring extends to G , a contradiction.

Hence $\delta(G) \geq 5$. \square

2. THE REDUCED DUAL

Write G' for the dual of G : since G is a triangulation, G' is a cubic plane graph in which each vertex of G corresponds to a face of G' , each face of G to a vertex of G' , and each edge to a dual edge. A vertex of G of degree k corresponds to a k -gonal face of G' .

By Lemma 1.4, $\delta(G) \geq 5$, and Euler's formula gives $\sum_{u \in V(G)} (6 - \deg u) = 12$, so G has a vertex of degree exactly 5 (indeed at least twelve). Fix such a vertex v . Its dual face F_v is a pentagon, bounded by the five dual vertices corresponding to the five faces of G incident to v .

Definition 2.1 (Reduced dual). Let v be a degree-5 vertex of G with pentagonal dual face F_v , and fix an index $i \in \{0, 1, 2, 3, 4\}$. The *reduced dual* $\hat{G}'_{v,i}$ is the plane graph obtained from G' as follows.

- (1) Delete the five dual vertices on the boundary of F_v , together with all edges incident to them. Each deleted vertex is cubic, with two edges on ∂F_v and one edge leaving F_v ; deleting the five boundary vertices therefore removes the five external edges as well, dropping their five outer endpoints from degree 3 to degree 2. These five degree-2 vertices lie on the boundary of a single face F of the resulting graph.
- (2) List the five degree-2 vertices in clockwise order around F as $A = (A_0, A_1, A_2, A_3, A_4)$.
- (3) Add a new vertex v_n and join it to A_i, A_{i+1} , and A_{i+2} (indices mod 5) by three new edges.
- (4) Add a new edge between A_{i+3} and A_{i+4} (indices mod 5).

Remark 2.2. Steps (3) and (4) restore cubicity: A_i, A_{i+1}, A_{i+2} each gain one edge to v_n and A_{i+3}, A_{i+4} each gain the new edge, so all five return to degree 3, and v_n has degree 3. Since A_i, \dots, A_{i+2} and A_{i+3}, A_{i+4} are each consecutive along ∂F , the new vertex and edge can be drawn inside F without crossings, so $\hat{G}'_{v,i}$ is again a cubic plane graph. The construction depends on the choice of i up to the rotational symmetry of A .

Definition 2.3 (Edges of the reduced dual). The four edges added in steps (3) and (4) of Definition 2.1 are named as follows. The chord $A_{i+3}A_{i+4}$ is the *merged edge*; the edge $A_{i+1}v_n$ is the *spike edge*; the edge $A_i v_n$ is the *side-0 edge*; and the edge $A_{i+2}v_n$ is the *side-1 edge*. In the $i = 0$ case of Figure 1 these are $\{A_3, A_4\}$, $\{A_1, v_n\}$, $\{A_0, v_n\}$, and $\{A_2, v_n\}$ respectively.

We will use the following structural fact about proper 3-edge-colourings near a pentagonal face of a cubic plane graph; it is stated for a generic such graph H , not specifically for the reduced dual.

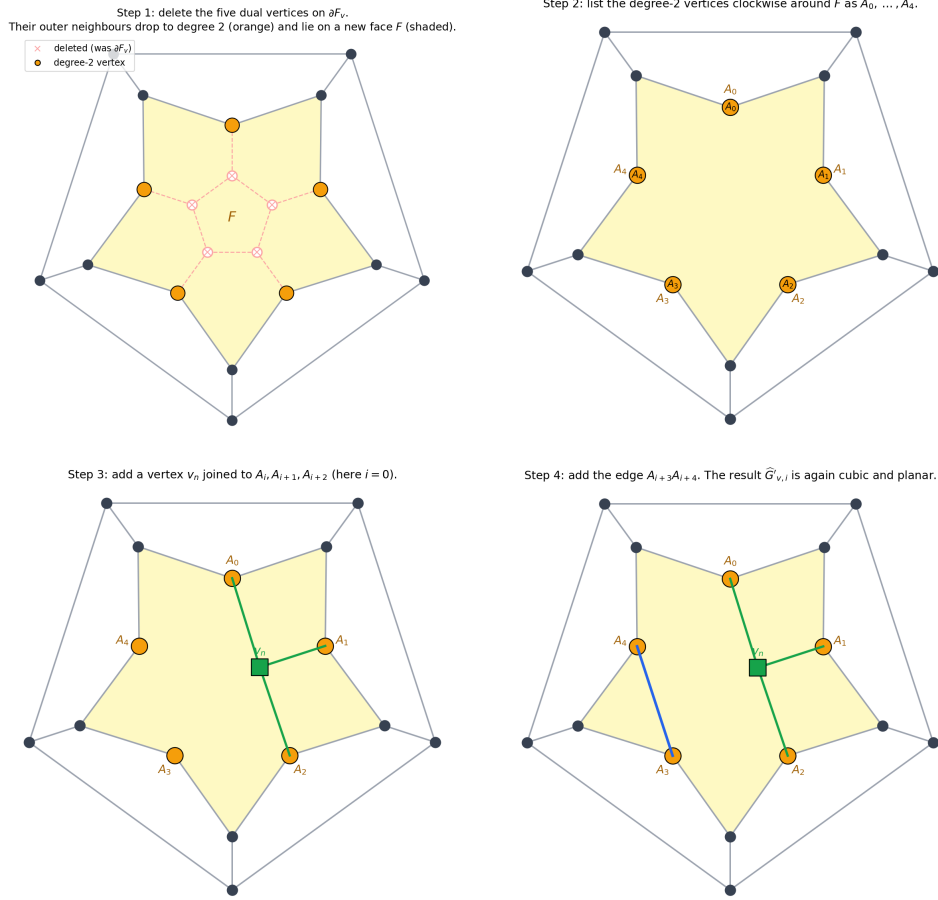


FIGURE 1. The four steps of Definition 2.1, illustrated on $G' =$ the dodecahedron (dual of the icosahedron) with F_v the inner pentagon and $i = 0$. Top left: delete the five boundary vertices of F_v , leaving five degree-2 vertices on a new face F . Top right: order them clockwise as A_0, \dots, A_4 . Bottom left: add v_n joined to A_0, A_1, A_2 . Bottom right: add the chord A_3A_4 , giving the cubic plane graph $\widehat{G}'_{v,0}$.

Lemma 2.4 (Pentagonal externals). *Let H be a cubic plane graph and F a pentagonal face of H , with ∂F traversed clockwise as u_0, u_1, u_2, u_3, u_4 . For each i let f_i be the unique edge of H incident to u_i that does not lie on ∂F . An assignment φ of colours from $\{1, 2, 3\}$ to the ten edges incident to $\{u_0, \dots, u_4\}$ is proper at every u_i if and only if there is some index j such that*

$$\varphi(f_j) = \varphi(f_{j+1}) = \varphi(f_{j+2}) \quad \text{and} \quad \{\varphi(f_{j+3}), \varphi(f_{j+4})\} = \{1, 2, 3\} \setminus \{\varphi(f_j)\},$$

indices mod 5.

Proof. Write $e_i = u_i u_{i+1}$ for the boundary edges of ∂F (indices mod 5). A colouring φ is proper at every u_i if and only if at each u_i the three incident edges e_{i-1}, e_i, f_i

receive three distinct colours; whenever this holds, $\varphi(f_i)$ is forced to be the unique colour in $\{1, 2, 3\} \setminus \{\varphi(e_{i-1}), \varphi(e_i)\}$, and φ restricts to a proper 3-edge-colouring of the cycle ∂F .

(\Rightarrow) The line graph of ∂F is C_5 , whose maximum independent set has size 2, so no colour appears more than twice on ∂F ; and since ∂F is an odd cycle, all three colours appear. The colour multiset on $(\varphi(e_0), \dots, \varphi(e_4))$ is therefore $(2, 2, 1)$, with the singleton at a unique position. Cyclically shifting indices we may place this position at 0; let c be the singleton colour. The remaining four edges form the path $e_1 e_2 e_3 e_4$, which by propriety alternates between the other two colours, so for some labelling $\{a, b, c\} = \{1, 2, 3\}$,

$$(\varphi(e_0), \varphi(e_1), \varphi(e_2), \varphi(e_3), \varphi(e_4)) = (c, a, b, a, b).$$

Reading off the forced values of $\varphi(f_i)$,

$$\varphi(f_0) = a, \quad \varphi(f_1) = b, \quad \varphi(f_2) = \varphi(f_3) = \varphi(f_4) = c,$$

which is the lemma's pattern at $j = 2$ (the cyclic shift maps this back to the corresponding j in the original indexing). This case is the unique proper 3-edge-colouring of ∂F up to cyclic shift and permutation of $\{1, 2, 3\}$ (since $5 \cdot 3! = 30 = P(C_5, 3)$, the chromatic polynomial of C_5 at 3), so it exhausts every proper φ .

(\Leftarrow) The lemma's hypothesis is invariant under cyclic shifts of indices and under permutations of $\{1, 2, 3\}$, so we may assume $j = 2$, $\varphi(f_2) = \varphi(f_3) = \varphi(f_4) = c$, $\varphi(f_0) = a$, and $\varphi(f_1) = b$, with $\{a, b, c\} = \{1, 2, 3\}$. Propriety at u_i and u_{i+1} requires $\varphi(e_i) \notin \{\varphi(f_i), \varphi(f_{i+1})\}$, which gives

$$\varphi(e_0) = c, \quad \varphi(e_1) = a, \quad \varphi(e_2) \in \{a, b\}, \quad \varphi(e_3) \in \{a, b\}, \quad \varphi(e_4) = b.$$

The remaining propriety condition $\varphi(e_{i-1}) \neq \varphi(e_i)$ holds automatically at u_0, u_1, u_4 , forces $\varphi(e_2) = b$ at u_2 , and then forces $\varphi(e_3) = a$ at u_3 . The resulting triples $(\varphi(e_{i-1}), \varphi(e_i), \varphi(f_i))$ at u_0, u_1, u_2, u_3, u_4 are

$$(b, c, a), \quad (c, a, b), \quad (a, b, c), \quad (b, a, c), \quad (a, b, c),$$

each a permutation of $\{1, 2, 3\}$, so φ is proper at every u_i . \square

Remark 2.5. The two-element condition $\{\varphi(f_{j+3}), \varphi(f_{j+4})\} = \{1, 2, 3\} \setminus \{\varphi(f_j)\}$ cannot be dropped: a 3-colouring satisfying $\varphi(f_j) = \varphi(f_{j+1}) = \varphi(f_{j+2})$ alone need not extend, e.g. $(1, 1, 1, 1, 2)$.

Since $\widehat{G}'_{v,i}$ is the dual of a triangulation on fewer vertices than G , it is 3-edge-colourable by the minimality of G . The following lemma constrains every such colouring.

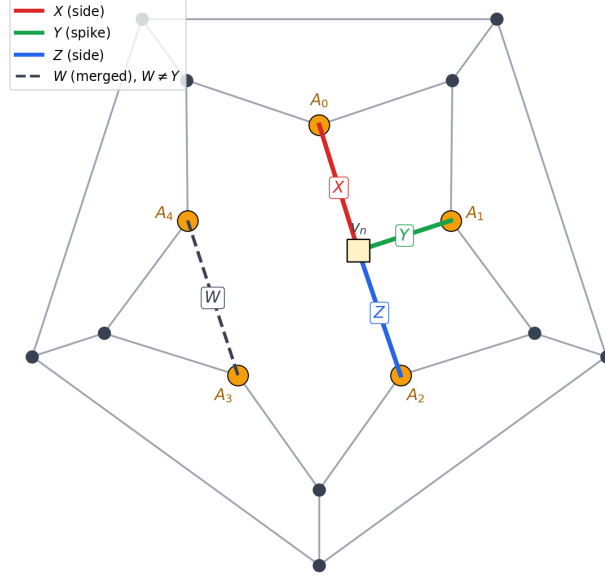
Lemma 2.6. *Let G be a minimal counterexample, and let $\widehat{G}'_{v,i}$ be a reduced dual of its dual G' . Then in every proper 3-edge-colouring of $\widehat{G}'_{v,i}$, the merged edge and the spike edge receive the same colour.*

Proof. After cyclically relabelling, assume $i = 0$. Suppose for contradiction that φ is a proper 3-edge-colouring of $\widehat{G}'_{v,0}$ in which the merged edge $\{A_3, A_4\}$ and the spike edge $\{A_1, v_n\}$ receive different colours (Figure 2, top), and write

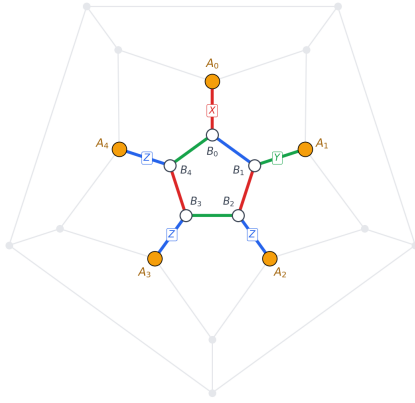
$$X = \varphi(\{A_0, v_n\}), \quad Y = \varphi(\{A_1, v_n\}), \quad Z = \varphi(\{A_2, v_n\}), \quad W = \varphi(\{A_3, A_4\}).$$

Propriety of φ at v_n forces $\{X, Y, Z\} = \{1, 2, 3\}$, and the assumption $W \neq Y$ leaves $W \in \{X, Z\}$.

Step 1: φ on $\widehat{G}_{v,0}$ assigns distinct colours X, Y, Z to the v_n -edges (propriety at v_n);
by hypothesis $W \neq Y$, forcing $W \in \{X, Z\}$.



Step 2: lift to G' when $W = Z$. The externals inherit $\psi(f) = (X, Y, Z, Z, X)$;
Lemma 2.4 colours the five edges of ∂F_v .



Step 3: lift to G' when $W = X$. The externals inherit $\psi(f) = (X, Y, Z, X, X)$;
Lemma 2.4 colours the five edges of ∂F_v .

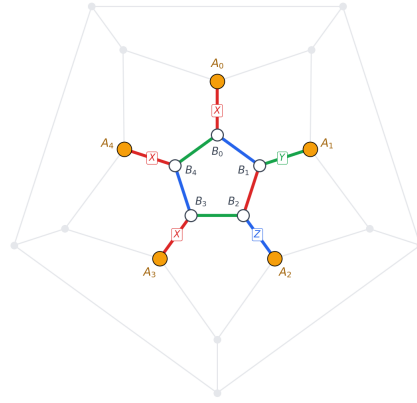


FIGURE 2. The proof of Lemma 2.6, illustrated for $i = 0$ on $G' =$ the dodecahedron. Top: under the assumption $W \neq Y$, propriety at v_n forces $W \in \{X, Z\}$. Bottom: in either case the lift to G' has externals satisfying the hypothesis of Lemma 2.4, which colours ∂F_v to extend ψ to a proper 3-edge-colouring of G' .

We lift φ to a colouring ψ of $E(G')$ as follows. Let B_0, \dots, B_4 be the boundary vertices of ∂F_v in G' , indexed so that $f_k = B_k A_k$. On every edge of G' that survived the reduction, set $\psi = \varphi$. At each A_k the two surviving edges retain their φ -colours, so the remaining edge at A_k — in G' this is the external f_k ; in $\widehat{G}'_{v,0}$ this is a v_n -edge ($k \in \{0, 1, 2\}$) or the chord ($k \in \{3, 4\}$) — is forced to take the third

colour at A_k . Since the two-surviving-edge colours at A_k agree in G' and $\widehat{G}'_{v,0}$, the third colour does too, giving

$$\psi(f_0) = X, \quad \psi(f_1) = Y, \quad \psi(f_2) = Z, \quad \psi(f_3) = \psi(f_4) = W$$

(the last two equalities holding because the chord is a single edge contributing its colour at each of A_3 and A_4).

It remains to assign colours to the five boundary edges $B_k B_{k+1}$ of ∂F_v . Apply Lemma 2.4 to G' at the face F_v with the B_k 's as its boundary vertices and the same indexing. The external vector $(\psi(f_0), \dots, \psi(f_4)) = (X, Y, Z, W, W)$ falls into one of two cases (Figure 2, bottom):

- if $W = Z$, it is (X, Y, Z, Z, Z) : three consecutive Z 's at positions 2, 3, 4, with $\{X, Y\} = \{1, 2, 3\} \setminus \{Z\}$;
- if $W = X$, it is (X, Y, Z, X, X) : three consecutive X 's at positions 3, 4, 0, with $\{Y, Z\} = \{1, 2, 3\} \setminus \{X\}$.

Each case satisfies the hypothesis of Lemma 2.4; its (\Leftarrow) direction therefore assigns colours to the boundary edges $B_k B_{k+1}$ that make ψ proper at every B_k .

The resulting ψ is a proper 3-edge-colouring of G' : proper at every B_k by the lemma, at every A_k by the construction, and at every other vertex because such a vertex has the same neighbourhood in G' as in $\widehat{G}'_{v,0}$ with the same incident-edge colours. By Tait's theorem, G' is 3-edge-colourable iff G is 4-vertex-colourable, contradicting that G is a counterexample. The assumption $W \neq Y$ is therefore false. \square

For a pair of colours $\{a, b\} \subseteq \{1, 2, 3\}$, the subgraph of $\widehat{G}'_{v,i}$ on the edges coloured a or b is 2-regular (since at each vertex exactly one of the three incident edges is excluded), and hence a disjoint union of cycles. We call each such cycle a $\{a, b\}$ -Kempe cycle, and reserve the notation for the specific cycle containing a given edge when the context makes it clear. Swapping the two colours on a single Kempe cycle yields another proper 3-edge-colouring of the same graph.

Lemma 2.7 (Kempe cycles through the spike). *Let G be a minimal counterexample, fix a reduced dual $\widehat{G}'_{v,i}$ of G' , and let φ be a proper 3-edge-colouring of $\widehat{G}'_{v,i}$. Write c for the common colour assigned by φ to the spike and the merged edge (Lemma 2.6), and c_0, c_1 for the colours of the side-0 and side-1 edges respectively, so $\{c, c_0, c_1\} = \{1, 2, 3\}$. Then*

- (1) *the $\{c, c_0\}$ -Kempe cycle through the spike edge contains both the side-0 edge and the merged edge;*
- (2) *the $\{c, c_1\}$ -Kempe cycle through the spike edge contains both the side-1 edge and the merged edge.*

Proof. We prove (1); (2) is the same argument with c_1 and the side-1 edge in place of c_0 and the side-0 edge.

The spike edge $\{A_{i+1}, v_n\}$ and the side-0 edge $\{A_i, v_n\}$ share the vertex v_n and receive the two colours c, c_0 , so they both lie on the $\{c, c_0\}$ -Kempe cycle through v_n . Suppose for contradiction that the merged edge lies on a different $\{c, c_0\}$ -Kempe cycle K (it lies on *some* such cycle, since it has colour c). Let φ' be obtained from φ by swapping the colours c and c_0 along K alone: this is a Kempe swap, so φ' is again a proper 3-edge-colouring of $\widehat{G}'_{v,i}$. Under φ' the spike edge — which is not on K — still has colour c , but the merged edge — which is on K — now has colour c_0 .

Hence in φ' the spike and the merged edge receive distinct colours, contradicting Lemma 2.6 applied to φ' . \square

3. AN ITERATED REDUCTION

The reduced-dual construction in Definition 2.1 can be iterated: starting from a proper 3-edge-colouring φ_1 of a reduced dual $\widehat{G}'_{v,i}$, we apply the construction again to that graph at a pentagonal face whose ten incident edges avoid the four named edges from the first reduction, extending φ_1 across the new reduction. The protected edges accumulate into a set E that grows by four per iteration, and the process terminates when E has blocked every pentagonal face.

Algorithm 3.1 (Iterated reduction with protected edges). Let G be a triangulation we assume to be a minimal counterexample to the Four Colour Theorem. The algorithm produces a sequence H_1, H_2, \dots of cubic plane graphs, proper 3-edge-colourings φ_t of H_t , and a growing set E of protected edges.

- (0) Form $G' := \text{dual}(G)$, a cubic plane graph.
- (1) Choose a degree-5 vertex v of G (equivalently a pentagonal face F_v of G') and an index $i_1 \in \{0, \dots, 4\}$. Apply Definition 2.1 to form $H_1 := \widehat{G}'_{v,i_1}$, and fix any proper 3-edge-colouring φ_1 of H_1 (one exists by the minimality of G).
- (2) Initialise $E := \{\text{spike}, \text{side-0}, \text{side-1}, \text{merged}\}$, the four named edges of the reduction in (1).
- (3) (Iterate.) At step $t \geq 2$, given H_{t-1} , φ_{t-1} , and $E \subseteq E(H_{t-1})$:
 - (a) Find a pentagonal face F of H_{t-1} whose ten incident edges — the five boundary edges of ∂F and the five external edges at ∂F — are all outside E . If no such F exists, terminate.
 - (b) By Lemma 2.4 applied to H_{t-1} at F under φ_{t-1} , the external vector has shape (a, b, c, c, c) up to cyclic rotation. Choose an index i_t for which $\varphi_{t-1}(f_{i_t+3}) = \varphi_{t-1}(f_{i_t+4})$ and $\varphi_{t-1}(f_{i_t}), \varphi_{t-1}(f_{i_t+1}), \varphi_{t-1}(f_{i_t+2})$ are three distinct colours.
 - (c) Apply Definition 2.1 to H_{t-1} at (F, i_t) to form H_t .
 - (d) Extend φ_{t-1} to a proper 3-edge-colouring φ_t of H_t : every surviving edge keeps its φ_{t-1} -colour, and each new edge takes the unique colour completing the palette at its endpoint (consistent across both endpoints of the chord by the choice of i_t).
 - (e) Add the four named edges of the step- t reduction to E .
- (4) Repeat (3) until termination.

Remark 3.2. At each iteration, $|V(H_t)| = |V(H_{t-1})| - 4$ and $|E(H_t)| = |E(H_{t-1})| - 6$, so H_t shrinks at a fixed rate; the protected set $|E|$ grows by exactly four; and every protected edge survives all subsequent reductions. Since the graph is finite, termination is guaranteed. By Lemma 2.4, step (b) never fails: some valid i_t always exists for any pentagonal face under any proper colouring. Termination is therefore combinatorial: it occurs precisely when E touches every pentagonal face of H_{t-1} .

Remark 3.3. Lemma 2.6 applies only at $t = 1$, when H_1 is a reduced dual of G' . For $t \geq 2$, H_t is a reduced dual of H_{t-1} rather than of G' , and H_{t-1} is itself 3-edge-colourable, so the non-3-edge-colourability argument that drives Lemma 2.6 does not carry over. Whether the constraints accumulated in E propagate any further structure to φ_t for $t \geq 2$ is left open.

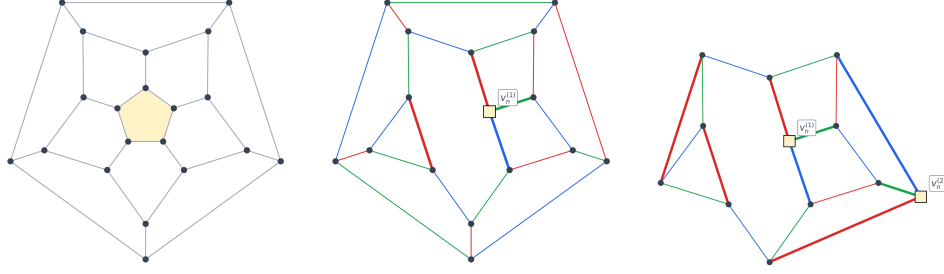


FIGURE 3. Algorithm 3.1 on $G' = \text{dodecahedron}$ (dual of the icosahedron). *Left:* G' (20 vertices, 30 edges), with F_v (the inner pentagon) shaded as the face chosen for the first reduction. *Centre:* H_1 (16 vertices, 24 edges) after step (1) with $i_1 = 0$, 3-edge-coloured by Sage; the four edges around $v_n^{(1)}$ in E are drawn thicker. *Right:* H_2 (12 vertices, 18 edges) after step (3) with $i_t = 0$; the only safe pentagonal face in H_1 was the outer pentagon, whose deletion produces $v_n^{(2)}$ and a second chord, giving eight protected edges. No safe pentagonal face remains, so the algorithm terminates. The generating script is `experiments/draw_iterated_reduction.py`.