Step 2 at $k = 9$ and $k = 12$: extending the adjacent-tire compatibility experiment

What this is

A continuation of `tire_fiber_step2.tex`: that note tested adjacent-tire compatibility (the chain-pigeonhole step) at $k \leq 6$, all 23/23 pairs compatible. This note pushes the experiment to $k \in \{9, 12\}$, enabled by the numpy-optimized fiber enumeration in `experiments/tire_fiber_chords_fast.py`. Together with the earlier data, every tested pair — now 23 at $k \leq 6$ plus 23 more at $k \in \{9, 12\}$, 46 in total — is compatible.

Script: `experiments/tire_fiber_step2_large.py`. Output: `experiments/tire_fiber_step2_large_data.t`

Setup recap

Two adjacent tires T_1, T_2 share a cycle γ of length k. We project each tire's σ-support onto γ:

$$S_1 := \pi_D^{(1)}(\mathcal{C}^{(1)}) \subseteq \{1, 2, 3\}^k, \quad S_2 := \pi_U^{(2)}(\mathcal{C}^{(2)}) \subseteq \{1, 2, 3\}^k,$$

and ask whether $S_1 \cap S_2 \neq \emptyset$ (forward), or $S_1 \cap \mathrm{reverse}(S_2) \neq \emptyset$ (reverse). Either non-empty intersection means the pair is *compatible*.

Chord configurations used

Under the Steiner-poor model, every O-face must have at most 3 B_{in} edges. For $k > 3$ this forces specific chord matchings. Constrained chord sets used (validated):

label	chord set	face sizes
`K9_CHORDS_3`	$\{(0, 2), (3, 5), (6, 8)\}$	$(2, 2, 2, 3)$
`K9_CHORDS_NESTED`	$\{(0, 2), (3, 7), (4, 6)\}$	$(2, 2, 2, 3)$ (different arrangement)
`K12_CHORDS_3`	$\{(0, 3), (4, 7), (8, 11)\}$	$(3, 3, 3, 3)$ (symmetric thirds)
`K12_CHORDS_NESTED`	$\{(0, 3), (4, 11), (5, 7), (8, 10)\}$	$(2, 2, 2, 3, 3)$

Results at $k = 9$ (13/13 compatible)

T_1	T_2	$\lvert S_1\rvert$	$\lvert S_2\rvert$	$\lvert S_1 \cap S_2\rvert$ fwd	rev	compat?
$(9, \texttt{K9-3}, \mathrm{SP})$	$(9, \texttt{K9-3}, \mathrm{SP})$	1296	7866	1188	858	yes
$(9, \texttt{K9-3}, \mathrm{SP})$	$(9, \texttt{K9-N}, \mathrm{SP})$	1296	7536	1242	1110	yes
$(9, \texttt{K9-N}, \mathrm{SP})$	$(9, \texttt{K9-N}, \mathrm{SP})$	1296	7536	1176	1224	yes
$(12, \texttt{K9-3}, \mathrm{SP})$	$(9, \texttt{K9-3}, \mathrm{SP})$	1296	7866	1188	858	yes
$(9, \texttt{K9-3}, \mathrm{SP})$	$(12, \texttt{K12-3}, \mathrm{SP})$	1296	1302	72	90	yes
$(9, \texttt{K9-3}, \mathrm{SP})$	$(12, \texttt{K12-N}, \mathrm{SP})$	1296	9456	840	606	yes
$(9, \texttt{K9-3}, \mathrm{SR})$	$(9, \texttt{K9-3}, \mathrm{SP})$	19683	7866	7866	7866	yes
$(9, \texttt{K9-3}, \mathrm{SP})$	$(3, -, \mathrm{SR})$	1296	3681	108	108	yes
$(9, \texttt{K9-3}, \mathrm{SP})$	$(4, (0, 2), \mathrm{SP})$	1296	3162	276	108	yes
$(9, \texttt{K9-3}, \mathrm{SP})$	$(6, (0, 3), \mathrm{SP})$	1296	942	54	60	yes
$(9, \texttt{K9-3}, \mathrm{SP})$	$(6, (0, 2)(3, 5), \mathrm{SP})$	1296	6210	402	324	yes
$(9, \texttt{K9-N}, \mathrm{SP})$	$(6, (0, 3), \mathrm{SP})$	1296	942	54	36	yes
$(9, \texttt{K9-N}, \mathrm{SP})$	$(12, \texttt{K12-N}, \mathrm{SP})$	1296	9456	732	768	yes

Universe size at $k = 9$: $3^9 = 19{,}683$.

Results at $k = 12$ (10/10 compatible)

T_1	T_2	$\lvert S_1\rvert$	$\lvert S_2\rvert$	$\lvert S_1 \cap S_2\rvert$ fwd	rev	compat?
$(12, \texttt{K12-3}, \mathrm{SP})$	$(12, \texttt{K12-3}, \mathrm{SP})$	1296	12840	960	564	yes
$(12, \texttt{K12-N}, \mathrm{SP})$	$(12, \texttt{K12-N}, \mathrm{SP})$	7776	100938	5928	3414	yes
$(12, \texttt{K12-3}, \mathrm{SP})$	$(12, \texttt{K12-N}, \mathrm{SP})$	1296	100938	1128	912	yes
$(12, \texttt{K12-3}, \mathrm{SP})$	$(3, -, \mathrm{SR})$	1296	31176	192	192	yes
$(12, \texttt{K12-3}, \mathrm{SP})$	$(4, (0, 2), \mathrm{SP})$	1296	27378	48	60	yes
$(12, \texttt{K12-3}, \mathrm{SP})$	$(6, (0, 3), \mathrm{SP})$	1296	9882	**6**	**6**	yes
$(12, \texttt{K12-3}, \mathrm{SP})$	$(6, (0, 2)(3, 5), \mathrm{SP})$	1296	61224	12	12	yes
$(12, \texttt{K12-3}, \mathrm{SP})$	$(9, \texttt{K9-3}, \mathrm{SP})$	1296	94116	18	90	yes
$(12, \texttt{K12-N}, \mathrm{SP})$	$(9, \texttt{K9-3}, \mathrm{SP})$	7776	94116	552	852	yes
$(12, \texttt{K12-3}, \mathrm{SR})$	$(12, \texttt{K12-3}, \mathrm{SP})$	531441	12840	12840	12840	yes

Universe size at $k = 12$: $3^{12} = 531{,}441$.

Observations

Observation (Still compatible everywhere)**.** 46 *of* 46 *tested pairs are compatible across* $k \in \{3, 4, 5, 6, 9, 12\}$ *(*23 *from the earlier note plus* 23 *new at* $k \in \{9, 12\}$*). No counterexample has been found.*

Observation (The S_3-orbit pattern persists at $k = 12$)**.** *The smallest tested intersection at* $k = 12$ *is again* exactly 6 elements*, occurring at* $T_1 = (12, \texttt{K12-3}, SP)$ *vs.* $T_2 = (6, (0, 3), SP)$*. Direct inspection shows the six elements are a single* S_3*-orbit of the canonical pattern*

$$(1, 2, 3, 2, 2, 1, 3, 3, 2, 3, 1, 1).$$

Decoded against **K12-3***'s face structure (faces* $\{0, 1, 2\}$*,* $\{4, 5, 6\}$*,* $\{8, 9, 10\}$*, and the outer face* $\{3, 7, 11\}$ *on the* B_{in} *edges):*

- *face* $\{0, 1, 2\}$*:* σ*-values* $(1, 2, 3)$ *– a permutation of* $\{1, 2, 3\}$.
- *face* $\{4, 5, 6\}$*:* σ*-values* $(2, 1, 3)$ *– a permutation.*
- *face* $\{8, 9, 10\}$*:* σ*-values* $(2, 3, 1)$ *– a permutation.*

- *face* $\{3,7,11\}$*: σ-values* $(2,3,1)$ *– a permutation.*

Every face receives all three colors exactly once. This is precisely the "Latin-square-flavoured" structural pattern from the $k=6$ worst case, now extended to $k=12$.

Observation (Bigger supports come from more chords)**.** *Nested chord sets give substantially larger supports than symmetric ones:*

- *At $k=12$, `K12-3` (symmetric, faces $3+3+3+3$) gives $|S_1| = 1296$.*
- *`K12-N` (nested, faces $2+2+2+3+3$) gives $|S_1| = 7776 = 6 \cdot 1296$.*

The factor of 6 *is suggestive but I have not chased it.*

Observation (Intersection sizes are multiples of 6)**.** *Every observed forward and reverse intersection size in the new data is a multiple of* 6*:*

$$1188, 1242, 1176, 858, 1110, 1224, 72, 90, 840, 606, 7866, 108, 276, 54, 60, 402, 324, 36, 732, 768, 960, 564, 5928, 3414, 1128, 912, 192, 48, 60, 6, 12, 18, 90, 552, 852, 12840$$

This is consistent with both S_1 and S_2 being S_3-invariant (closed under color permutations), so their intersection decomposes into S_3-orbits, each of size 6*.*

Speculative theorem

> **Conjecture.** For any SP-feasible tire T (i.e. every O-face has at most 3 B_{in} edges), the projection $\pi_D(\mathcal{C}(T))$ on the γ-side contains the "Latin-flavoured" subset
>
> $$\mathcal{L}(\gamma, O) \;:=\; \{\sigma \in \{1,2,3\}^{|\gamma|} : \sigma \text{ restricted to each } O\text{-face is a permutation of } \{1,2,3\}\},$$
>
> which is an S_3-invariant set of size at least 6 (and exactly 6 in the maximally constrained case). Adjacent tires share this common substructure on γ, so the chain-pigeonhole intersection is non-empty.

The data is consistent with this conjecture and points to a structural proof: any tire that admits an edge 3-coloring at all must admit a globally "Latin-style" one, and Latin-style colorings are dictated entirely by face structure (not by which tire's face it is). Adjacent tires that share a cycle γ see the same Latin constraints from each other's side, so their Latin-style supports agree.

This would be the analog, on the chord side, of step 1's "saturation iff $m \geq k$" result on the spoke-only side.

Performance notes

The numpy-optimized enumeration in `experiments/tire_fiber_chords_fast.py` replaces the brute-force 3^n-iteration with direct construction of the $2^n + 2(-1)^n$ proper edge 3-colorings of C_n. Speedups benchmarked against the slow version:

- $n=12$: 0.001s vs 0.21s – $\sim 146\times$
- $n=15$: 0.013s vs 5.4s – $\sim 424\times$
- $n=18$: 0.11s vs (extrapolated) $\sim$ 150s
- $n=24$: 6.6s vs (extrapolated) $\sim$ 30 hours

Total wall time for the $k=9$ block: a few seconds; $k=12$ block: $\sim$ 4 minutes (dominated by the single SR-vs-SP case at $n=24$).

Caveats

1. **Still finite, still not a proof.** 46 pairs is still a small slice. Larger k (≥ 15) or unusual chord configurations could harbour a counterexample. The conjecture above suggests there is no counterexample, but is unproven.

2. **Multi-tire chains.** Step 2 is pairwise compatibility. A long nested chain of SP tires requires pairwise overlap at each shared cycle *plus* mutual consistency across all shared cycles simultaneously. The Latin-style conjecture, if true, would imply chain-wide consistency via a common Latin coloring of all annular faces at once.

3. **Model still SP/SR only.** Intermediate sub-triangulations (some Steiner vertices, some not) are not enumerated.

4. **Chord set space is sampled, not enumerated.** Each k has many distinct chord matchings; this experiment uses only one or two per k. More exhaustive enumeration could strengthen the empirical evidence.