Edge 3-colorings of small outerplanar graphs with $\Delta \leq 3$: a menagerie

Setup

For a graph G write $P_e(G,k)$ for the number of proper k-edge-colorings of G (= the chromatic polynomial of the line graph $L(G)$ evaluated at k). Throughout this note $\Delta(G)$ denotes the maximum degree of G; we are interested in $k=3$ and $\Delta(G) \leq 3$, with G outerplanar.

There is no *universal* closed form for $P_e(G,3)$ on the class of subcubic outerplanar graphs, but the class is small enough that every G in it decomposes along its block–cut tree into building blocks each of which admits a closed-form count. The building blocks form a short menagerie.

The menagerie

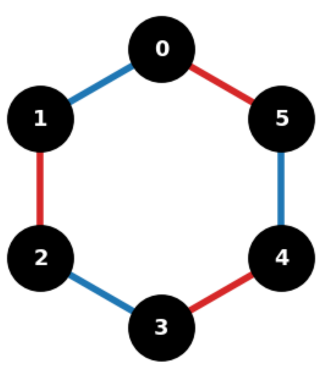
1. Path P_n (n vertices, $n-1$ edges; $\Delta \leq 2$)

The line graph $L(P_n)$ is the path P_{n-1}, so

$$P_e(P_n,k) \;=\; P_{\text{vert}}(P_{n-1},k) \;=\; k(k-1)^{n-2}, \qquad P_e(P_n,3) \;=\; 3 \cdot 2^{n-2}.$$

(For a single edge, $n=2$, the count is 3; for two edges in a row, $n=3$, the count is 6.)

2. Cycle C_n ($\Delta = 2$)

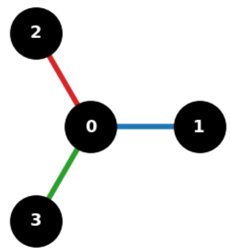


The line graph $L(C_n) = C_n$, so

$$P_e(C_n, k) \;=\; (k-1)^n + (-1)^n (k-1), \qquad P_e(C_n, 3) \;=\; 2^n + 2(-1)^n.$$

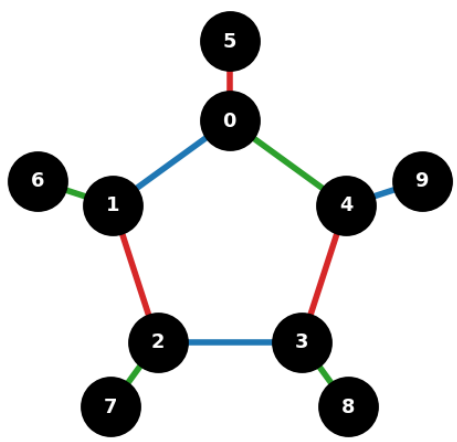
For even n the count is $2^n + 2$; for odd n it is $2^n - 2$.

3. Star $K_{1,3}$ (a single $\Delta = 3$ vertex)



Three pairwise-incident edges at a single vertex must carry three distinct colors, so $P_e(K_{1,3}, 3) = 3! = 6$. More generally $P_e(K_{1,d}, k) = k(k-1)\cdots(k-d+1)$, which is positive iff $k \geq d$, i.e. iff $k \geq \Delta(G)$.

4. Corona $C_n \circ K_1$ (cycle with one leaf per cycle vertex; $\Delta = 3$)



Each cycle vertex v has degree 3 in $C_n \circ K_1$: its two cycle edges must carry distinct colors and its leaf must carry the unique remaining third color. So the leaf coloring is *forced* by the cycle coloring, and

$$P_e(C_n \circ K_1, 3) \;=\; P_e(C_n, 3) \;=\; 2^n + 2(-1)^n.$$

This is the form of the partial tire dual $D(T)$ in the spoke-only case (with $L = n + m$).

5. Trees with $\Delta \leq 3$

A tree T on n vertices has $|E(T)| = n - 1$ edges; its line graph $L(T)$ is a *block graph* (every block is a clique). Edge-color a tree greedily by processing edges in BFS order from a leaf: when an edge $\{u, v\}$ is added, the only colors forbidden are those already used on the edges incident to its already-colored endpoint. Hence at any vertex of degree d, when the d-th edge is added there are exactly $k - (d - 1)$ choices. For $k = 3$ and $\Delta \leq 3$:

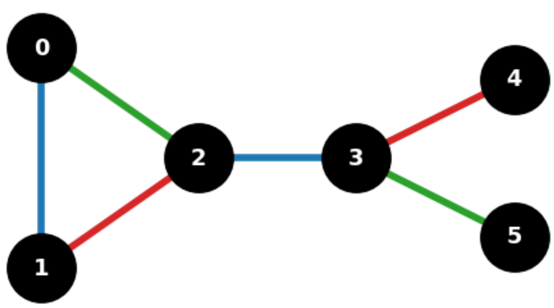
$$P_e(T, 3) \; = \; 3 \prod_{e \in E(T) \setminus \{e_0\}} (3 - d_e),$$

where e_0 is the first edge processed and d_e is the number of already- processed edges incident to the new endpoint of e (between 1 and 2, since $\Delta \leq 3$). In practice this gives a clean product depending only on the degree sequence of T.

6. Two-connected outerplanar with $\Delta = 3$ is just C_n

The only 2-connected outerplanar graphs are polygons (with optional chords). Each chord adds a degree to each of its two endpoints; if every vertex on the polygon C_n already has degree 2 from the cycle, then we may add at most one chord-endpoint per vertex, so chords must form a matching. Already a *single* chord on a 2-connected outerplanar graph forces both endpoints to degree 3. In particular, the only 2-connected outerplanar graph with $\Delta \leq 3$ in which the maximum is actually attained at *every* vertex would be a polygon with a perfect matching of chords; but each chord crosses some other (mod the planar embedding) unless the two matched vertices are adjacent on the polygon, which collapses the "2-connected" assumption. The upshot is: *the 2-connected blocks in our class are just cycles.*

7. Block–cut decomposition

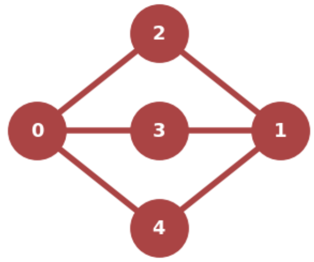


A general subcubic outerplanar graph is a union of cycle-blocks and edge-blocks glued at cut vertices. By case 6 each 2-connected block is a cycle; the remaining blocks are single edges (i.e. tree edges). At a cut vertex v of degree $d_v \in \{2, 3\}$, the colors of the d_v edges incident to v must be pairwise distinct. Counting $P_e(G, 3)$ for the whole graph G amounts to counting colorings of each block independently and then enforcing the distinct-color constraint at every cut vertex. For $k = 3$ and $\Delta \leq 3$ this gives

$$P_e(G, 3) \; = \prod_{B \text{ cycle block of } G} P_e(B, 3) \; \cdot \prod_{B \text{ edge block of } G} P_e(B, 3) \; / \prod_{v \text{ cut vertex}} (\text{normalization at } v),$$

where the normalization corrects the over- or under-counting at the cut-vertex constraint. For each cycle-block $B = C_n$ contributing $2^n + 2(-1)^n$ proper 3-edge-colorings, and each edge-block contributing 3, this product is computable in time linear in $|V(G)| + |E(G)|$.

Outside the menagerie: theta graphs



The complete bipartite graph $K_{2,3}$, equivalently the theta graph $\theta(2, 2, 2)$, is *not* outerplanar (it is a forbidden minor for outerplanarity). In our tire-graph application this is the structure of the interior dual subgraph of $D(T)$ when the inner outerplanar graph O has a bridge: two trivalent vertices d_f connected by three internally vertex-disjoint paths. Such a $D(T)$ falls outside the simple block-cut menagerie above and its $P_e(\cdot, 3)$ does not reduce to a product over cycle-blocks; instead it is computed directly by deletion–contraction on the theta-graph structure, or via a transfer matrix on the three paths.