NESTED TIRE DECOMPOSITIONS OF PLANE TRIANGULATIONS

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Abstract. We establish the foundational structure of nested level-induced tire decompositions of a plane triangulation G. A *level source* of G induces a BFS layering of G and endows the inner planar dual G' with a *dual depth* grading. The basic object of study is the *tire graph* T — a plane graph whose outer and inner boundaries bound a closed planar region, the *tire tread* R, triangulated by the *annular edges* E_{ann}. We define medial tire graphs and prove a basic colour-count bound for their annular medial cycle. Our main structural results are the *tire-component lemma*, the *tire-tread partition theorem*, and the rooted *tire-tree decomposition*, which together organize the bounded faces of G into nested tire treads.

1. Introduction

A classical theorem of Tait recasts the Four Colour Theorem in dual, edge-colouring terms: a plane triangulation G is properly 4-vertex-colourable if and only if its dual cubic graph G' is properly 3-edge-colourable. Thus a minimal counterexample to the Four Colour Theorem – a smallest triangulation admitting no proper 4-colouring – corresponds to a smallest cubic plane graph admitting no proper 3-edge-colouring.

The structural study of such a minimal counterexample is the overarching motivation for the present line of work. This first paper establishes the foundational vocabulary — level sources, dual depth, tire graphs, and partial tire duals — on which subsequent papers in the series build. In particular, the companion paper [3] uses these definitions to develop nested-cycle structure theorems and chain-pigeonhole conjectures for tire annular subgraphs of G'.

Related work. The structural object underlying this programme — the set of proper 4-colourings of a boundary cycle that extend to a colouring of a bounded planar region — is classical. Birkhoff's reducibility analysis of the diamond configuration [4] is the earliest instance of computing such extension sets to attack the Four Colour Theorem; the chromatic polynomial framework of Birkhoff and Lewis [5] systematised the counting. Tutte studied how the chromatic polynomial of a rooted planar triangulation decomposes along its outer boundary [8] and developed an algebraic theory of graph colourings organised around separating subgraphs [7, 6]. The most recent and structurally closest parallel is Dvořák and Lidický's analysis of *coloring count cones* [11], which characterises the possible boundary-extension functions on a fixed outer cycle of a near-triangulation. The Heesch–Appel–Haken approach [9, 10]

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also uses boundary-extension reasoning, but case-by-case on a finite unavoidable set of local configurations rather than as part of a global structural induction.

The tire-tree decomposition introduced here differs from each of these in shape rather than ingredients. Birkhoff, Tutte, and Dvořák–Lidický all study *one* boundary; Heesch and the cleaned-up Appel–Haken proof [10] study a finite collection of local boundaries. The present framework organises the entire triangulation into a hierarchy of annular regions glued along level cycles, and asks whether boundary-extension constraints compose compatibly up the hierarchy. To the authors' knowledge, no prior work on the Four Colour Theorem has been organised around a global nested-cycle decomposition of this kind.

Throughout, $G = (V, E)$ is a plane maximal planar graph (a triangulation) with a fixed planar embedding Π_G. We write $|V| = n$, so $|E| = 3n - 6$ and G has $2n - 4$ triangular faces.

Definition 1.1 (Level source)**.** A *level source* of G is a set $S \subseteq V$ that is either

- a single vertex $\{v\}$ (a *vertex source*), or
- a set inducing a simple cycle in G — i.e. $G[S]$ is a simple cycle of length ≥ 3 (a *cycle source*).

Definition 1.2 (Levels)**.** Given a level source $S \subseteq V$, the *level* of $v \in V$ is $\ell_G(v) = \mathrm{dist}_G(v, S)$, the graph distance from v to the nearest source vertex. We write $L_d := \{v \in V : \ell_G(v) = d\}$ for the *level-d vertex set*, and abbreviate $L_{<d} := \bigcup_{d' < d} L_{d'}$ and $L_{\geq d} := \bigcup_{d' \geq d} L_{d'}$ (similarly $L_{>d}$, $L_{\leq d}$).

Definition 1.3 (Dual)**.** The *dual* of G, written G', is the inner (weak) planar dual of G with respect to the embedding Π_G: it has one vertex d_f for each bounded face f of G, and an edge joining d_f and $d_{f'}$ for each edge of G shared by two bounded faces f and f'. The unbounded outer face contributes no vertex, and edges of G on the outer boundary contribute no dual edge. Since G is a triangulation, each vertex $d_f \in V(G')$ corresponds to a triangular face f of G, and we write $V(f) \subseteq V$ for its three incident vertices.

Definition 1.4 (Dual depth)**.** Given a level source $S \subseteq V$, the *dual depth* of a dual vertex $d_f \in V(G')$ is

$$\delta_G(d_f) = \min_{v \in V(f)} \ell_G(v) = \min_{v \in V(f)} \mathrm{dist}_G(v, S),$$

the smallest level among the three vertices of G bounding the face f.

Definition 1.5 (Depth-d dual subgraph and its components)**.** For $d \geq 0$, the *depth-d dual subgraph* is

$$G'_d \;:=\; G'\big[\,\{d_f \in V(G') : \delta_G(d_f) = d\}\,\big],$$

the inner-dual subgraph induced on the dual vertices of dual depth d. For a connected component C' of G'_d we write

$$F_{C'} := \{f : d_f \in V(C')\}, \qquad V_{C'} := \bigcup_{f \in F_{C'}} V(f),$$

for its set of faces and the vertices of G bounding them, and $R_{C'} := \bigcup_{f \in F_{C'}} f \subseteq |\Pi_G|$ for the closed planar region these faces cover.

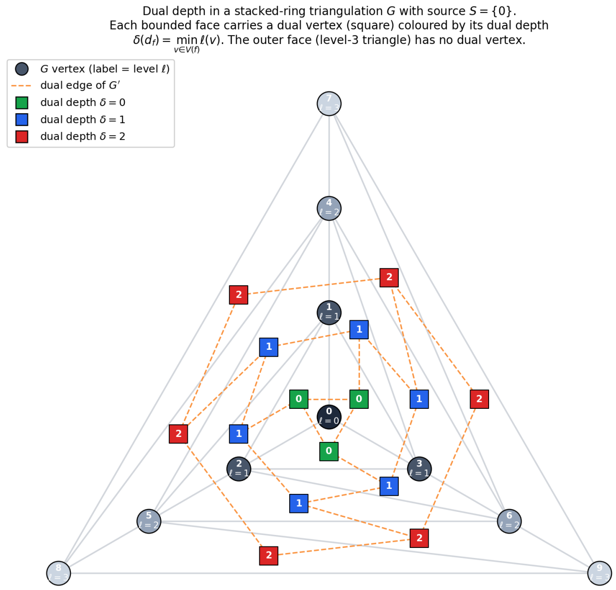

FIGURE 1. Dual depth in a stacked-ring triangulation G with level source $S = \{0\}$. Each G vertex is labelled by its level ℓ. Each bounded face carries a dual vertex (square, joined by dashed dual edges) coloured by its dual depth $\delta(d_f) = \min_{v \in V(f)} \ell(v)$: the central fan has depth 0, the inner annulus depth 1, and the outer annulus depth 2. The outer face (the level-3 triangle) is excluded from the inner dual and carries no dual vertex.

Definition 1.6 (Tire graph)**.** A *tire graph* consists of a plane graph T together with an *outer boundary* $B_{\text{out}} \subseteq T$ and a *connected inner outerplanar graph* $O \subseteq T$ with $V(B_{\text{out}}) \cap V(O) = \emptyset$, where

- B_{out} is either a simple cycle of length ≥ 3 or a single vertex (a *degenerate outer boundary*);
- O is a connected outerplanar graph; its *inner boundary* B_{in} is the closed walk in O that traces the boundary of O's outer face in the inherited embedding, which is a simple cycle when O is 2-connected and a non-simple closed walk in general (visiting bridges twice and cut-vertices multiple times); if $|V(O)| = 1$, we say T has a *degenerate inner boundary*.

At most one of $B_{\text{out}}, B_{\text{in}}$ may be degenerate. The vertex and edge sets of T are

$$V(T) = V(B_{\text{out}}) \cup V(O), \qquad E(T) = E(B_{\text{out}}) \cup E(O) \cup E_{\text{ann}},$$

where E_{ann} — the *annular edges* — has the property that, in the plane embedding of T, the closed planar region R bounded externally by B_{out} and internally by B_{in} is partitioned into triangular faces of T whose union is R. We call R the *tire tread* of T and write F_{ann} for this set of triangular faces (the *annular faces*).

When B_{out} is a simple cycle and O is 2-connected, the tread is a closed annulus. More generally, R is a closed planar region that may fail to be a 2-manifold at cut-vertices of O (where two "lobes" of the depth-d region meet at a single vertex);

the inner boundary B_{in} is then a non-simple closed walk that visits the cut-vertex multiple times. The relaxed definition accommodates outerplanar inner graphs with bridges or cut-vertices, while O itself remains connected. When either boundary is degenerate, the tread is a closed disk with that vertex as apex.

We summarize the data of a tire graph as the triple $T = (B_{\text{out}}, O, E_{\text{ann}})$, from which B_{in}, the annular faces F_{ann}, and the tread R are determined; we freely identify a tire graph with its underlying plane graph T.

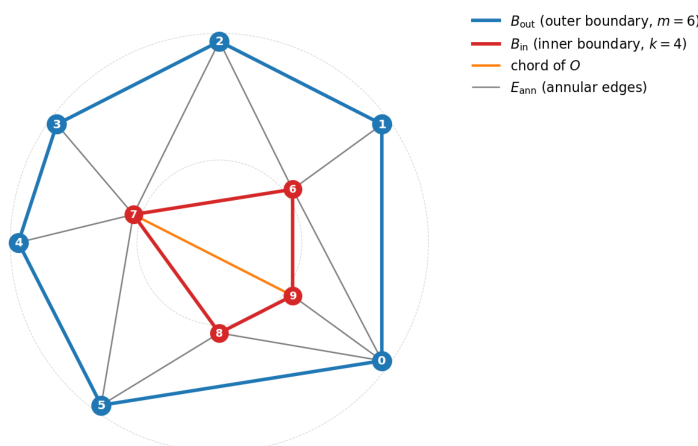

FIGURE 2. A tire graph with non-degenerate boundaries: outer boundary B_{out} a 6-cycle on vertices $0, \dots, 5$ (blue), inner boundary B_{in} a 4-cycle on vertices $6, \dots, 9$ (red), inner outerplanar graph $O = B_{\text{in}} \cup \{7\text{–}9\}$ (with one chord, orange), and E_{ann} (grey) tiling the annulus between B_{out} and B_{in} by ten triangular faces.

Definition 1.7 (Medial tire graph)**.** Let $T = (B_{\text{out}}, O, E_{\text{ann}})$ be a tire graph. Let T° be the plane graph obtained from T by deleting every edge of O that is not on the inner boundary walk B_{in}. Equivalently, T° keeps B_{out}, B_{in}, and the annular edges, but omits the chords of the inner outerplanar graph O.

The *medial tire graph* of T, denoted $M_{\text{tire}}(T)$, is obtained from the medial graph $M(T^\circ)$ by deleting every medial edge whose two endpoint-vertices correspond either to two edges of B_{out} or to two edges of B_{in}. Thus $V(M_{\text{tire}}(T))$ is naturally indexed by the non-chord edges of T, and two such vertices are adjacent exactly when the corresponding edges are consecutive on the boundary of a face of T°, except for consecutive pairs lying wholly along one of the two boundary walks. The medial vertices corresponding to edges of E_{ann} are called the *annular medial vertices.*

Theorem 1.8 (Annular medial colour bound)**.** *Let $T = (B_{\text{out}}, O, E_{\text{ann}})$ be a tire graph with non-degenerate boundaries and simple inner boundary B_{in}. Let $A(T)$ be the subgraph of $M_{\text{tire}}(T)$ induced by the annular medial vertices. For a graph H, write $\mathrm{Col}_3(H)$ for the set of proper 3-vertex-colourings of H. Then $A(T)$ is a cycle and*

$$|\,\mathrm{Col}_3(M_{\text{tire}}(T))| \;\le\; |\,\mathrm{Col}_3(A(T))|.$$

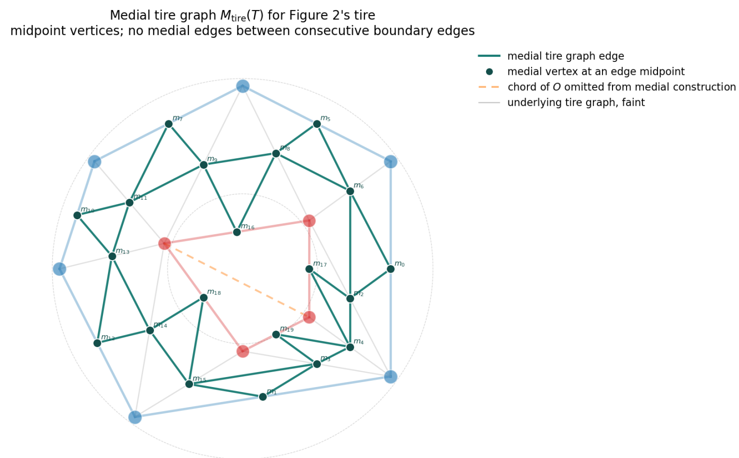

FIGURE 3. The medial tire graph for the tire in Figure 2. The chord of O is drawn faintly and omitted before taking the medial graph; medial edges between consecutive outer-boundary edges or consecutive inner-boundary edges are also omitted. Each medial vertex is placed at the midpoint of its corresponding retained tire edge.

Proof. Since the tread is a triangulated annulus with no vertices in its interior, each annular face has exactly one boundary edge, lying either on B_{out} or on B_{in}, and exactly two annular edges. As the annular faces are traversed cyclically around the tread, consecutive faces share one annular edge. Equivalently, the annular edges occur in a cyclic order in which each annular face contains two consecutive annular edges. Hence the subgraph of $M_{\mathrm{tire}}(T)$ induced by the annular medial vertices is a cycle.

Consider the restriction map from proper 3-colourings of $M_{\mathrm{tire}}(T)$ to colourings of this annular medial cycle $A(T)$. We claim that this map is injective. Let x be a non-annular medial vertex. Then x corresponds to an edge of B_{out} or B_{in}, since the chords of O were omitted before forming $M_{\mathrm{tire}}(T)$. This boundary edge is incident to a unique annular face of T°, and the other two edges of that face are annular edges. Therefore x is adjacent in $M_{\mathrm{tire}}(T)$ to the two annular medial vertices corresponding to those two annular edges.

Those two annular medial vertices are adjacent to each other, because their annular edges are consecutive on the same triangular annular face. In any proper 3-colouring they therefore receive two distinct colours, and x is forced to receive the remaining third colour. Thus every non-annular medial vertex has its colour uniquely determined by the colouring of $A(T)$. Two colourings of $M_{\mathrm{tire}}(T)$ with the same restriction to $A(T)$ are identical, so the restriction map is injective. The stated inequality follows. $\square$

Remark 1.9. Let $\mu = |V(B_{\mathrm{out}})|$ and $\nu = |V(B_{\mathrm{in}})|$. By Euler's formula on the tire tread R, the tire graph has $\mu + \nu$ triangular faces inside R and $|E_{\mathrm{ann}}| = \mu + \nu$ annular edges when neither boundary is degenerate; when exactly one boundary is degenerate (so $\min(\mu, \nu) = 1$), there are $\mu + \nu - 1$ triangular faces and $|E_{\text{ann}}| = \mu + \nu - 1$.

Proposition 1.10 (Source-side simple-cycle property)**.** *Let G be a maximal planar graph with planar embedding Π_G and single-vertex source v_0. Let $d \geq 1$, $v \in L_d$, and let C' be a connected component of G'_d such that v is incident to some face in $F_{C'}$. Then the depth-d faces in $F_{C'}$ incident to v form a single contiguous arc in v's rotation in Π_G.*

Equivalently: for any such component, the source-side boundary of $R_{C'}$ is a simple cycle in L_d (no cut-vertices at level d).

Proof. Suppose for contradiction that the depth-d faces in $F_{C'}$ at v lie in two or more disjoint arcs of v's rotation. Adjacent vertices in G differ in level by at most 1, so a face at v has depth exactly d iff both other vertices have level $\geq d$, and depth $\leq d-1$ iff at least one has level $d-1$. Hence the gaps between the depth-d arcs at v are populated by level-$(d-1)$ neighbours of v, occurring in at least two disjoint arcs of v's rotation. Pick p in one such gap and q in another.

The BFS ball $G[L_{<d}]$ is connected, so there exists a simple path P in $G[L_{<d}]$ from p to q. Define the closed walk

$$W \;:=\; v \to p \to P \to q \to v.$$

Every vertex of P lies in $L_{<d}$, while $\ell(v) = d$, so v is distinct from every vertex of P; P is simple, so its internal vertices are distinct; and $p \neq q$ since they lie in different gaps. Hence W is a simple cycle in G.

By the Jordan curve theorem, the planar embedding of W divides Π_G into two regions. In v's rotation, the edges $v - p$ and $v - q$ lie at two specific positions, and they split the rotation into two arcs; each arc lies in one of the two regions determined by W. By choice of p, q, the two arcs of depth-d faces at v in $F_{C'}$ lie in different regions of W (i.e., one arc on each side).

Since C' is connected in G' and contains depth-d faces in both arcs, there is a dual path $f_1, f_2, \ldots, f_k$ in G'_d with $f_1, f_k \in F_{C'}$ incident to v in different arcs, and with the intermediate faces $f_2, \ldots, f_{k-1}$ not incident to v (a shortest such dual path). Consecutive faces f_i, f_{i+1} share an edge e_i of G; for $i \geq 2$, both endpoints of e_i lie in $L_{\geq d}$ (since neither f_i nor f_{i+1} is incident to v, all six vertices of these two triangles lie in $L_{\geq d}$). In particular, e_i shares no endpoint with W except possibly v — and v is excluded from $f_2, \ldots, f_{k-1}$.

A planar edge with neither endpoint on a simple closed planar curve W has both of its incident faces on the same side of W. Applying this to each e_i ($i \geq 2$) inductively: starting from f_2 on the same side of W as f_1 (their shared edge $e_1 = w - w'$ opposite to v in f_1 has $w, w' \in L_{\geq d}$ and hence is not on W), the path $f_2 \to f_3 \to \cdots \to f_{k-1} \to f_k$ stays on one side of W.

But f_1 and f_k lie on different sides of W (by construction), contradicting the conclusion that the entire path lies on one side. □

Lemma 1.11 (Tire-component lemma)**.** *Let G be a maximal planar graph and let $S \subseteq V(G)$ be a level source. Fix a plane embedding Π_G of G in which S lies on the outer face (such an embedding exists for any planar graph and any single-vertex source). For $d \geq 0$, let C' be a connected component of the depth-d dual subgraph G'_d, with faces $F_{C'}$, bounding vertices $V_{C'}$, and region $R_{C'}$ as in Definition 1.5; let $T_{C'} := G[V_{C'}]$ inherit its embedding from Π_G.*

Then $T_{C'}$, with the inherited embedding, is a tire graph in the sense of Definition 1.6. Its outer boundary B_{out} is the side of $R_{C'}$ closer to S in Π_G, namely the level-d subgraph $G[V_{C'} \cap L_d]$ (a simple cycle or single vertex); its connected inner outerplanar graph is $O = G[V_{C'} \cap L_{d+1}]$, and its inner boundary B_{in} is the outer-face boundary closed walk of O in the inherited embedding (a simple cycle when O is 2-connected, a non-simple closed walk in general). The triangular faces of $T_{C'}$ inside the closed boundary region are exactly the faces of G in $F_{C'}$.

Proof. Outerplanarity of the two level parts. By construction S lies on the outer face of Π_G, so the outerplanarity lemma of [2] applies directly with (G, Π_G, S), giving that $G[L_{d'}]$ is outerplanar for each $d' \geq 0$. Subgraphs of outerplanar graphs are outerplanar, so $G[V_{C'} \cap L_d]$ and $G[V_{C'} \cap L_{d+1}]$ are both outerplanar.

Layer containment. Each $f \in F_{C'}$ has at least one vertex at level d, and adjacent vertices in G differ in level by at most 1; combined with $\delta_G(d_f) = d$, this forces $V(f) \subseteq L_d \cup L_{d+1}$. Hence $V_{C'} \subseteq L_d \cup L_{d+1}$, and $T_{C'}$ has vertex partition $V_{C'} = (V_{C'} \cap L_d) \sqcup (V_{C'} \cap L_{d+1})$.

Boundary edges are monochromatic in level. Each edge e on $\partial R_{C'}$ separates a face $f \in F_{C'}$ from a face $f' \notin F_{C'}$. Because f and f' share the edge e, their dual vertices are adjacent in G'; if both had depth d they would lie in the same component of G'_d, contradicting $d_f \in C'$ and $d_{f'} \notin C'$. Hence $\delta_G(d_{f'}) \neq d$; combined with the bounded-step property of δ across G'-adjacent faces, $\delta_G(d_{f'}) \in \{d-1, d+1\}$.

- If $\delta_G(d_{f'}) = d-1$, the third vertex w of $f' = \{u, v, w\}$ (where u, v are the endpoints of e) has $\ell(w) = d-1$. Each of u, v has $\ell \in \{d, d+1\}$ (from $V(f) \subseteq L_d \cup L_{d+1}$) and is adjacent to w, forcing $\ell(u), \ell(v) \in \{d-2, d-1, d\} \cap \{d, d+1\} = \{d\}$.
- If $\delta_G(d_{f'}) = d+1$, then all three vertices of f' lie in $L_{\geq d+1}$, so in particular $\ell(u) = \ell(v) = d+1$.

Each connected boundary component thus carries a single type at every edge: any vertex on a boundary component has two boundary edges incident to it (by R1, see below), both of the same type, so its level is fixed. Therefore each boundary component of $\partial R_{C'}$ is monochromatic in level.

Boundary structure. Each connected component of $\partial R_{C'}$ traces a closed walk in G that, by the monochromaticity above, lies entirely in L_d or entirely in L_{d+1}. By Proposition 1.10, the depth-d faces of $F_{C'}$ at any $v \in L_d \cap V_{C'}$ form a single contiguous arc in v's rotation, so the source-side boundary walk visits each L_d-vertex of $V_{C'}$ exactly once: it is a simple cycle. At vertices $v \in L_{d+1} \cap V_{C'}$ the depth-d faces may split into multiple arcs of v's rotation; this corresponds exactly to v being a cut-vertex of O, and the inner-side boundary walk visits v correspondingly many times — which is already accommodated by Definition 1.6 (where B_{in} is the outer-face boundary closed walk of O, not necessarily a simple cycle).

Outer boundary. Because S lies on the outer face of Π_G, the boundary curve(s) of $R_{C'}$ on the L_d side are closer to S in the embedding. In the inherited embedding of $T_{C'}$, the unique unbounded face is the merged region containing the rest of Π_G outside $R_{C'}$ on the S side, so its boundary — a simple cycle on L_d (or a single vertex when $V_{C'} \cap L_d = \{v_0\}$, the $d = 0$ case) — serves as B_{out}. We set $B_{\mathrm{out}} := G[V_{C'} \cap L_d]$ if this is a cycle, and the single vertex $\{v_0\}$ in the degenerate case.

Inner outerplanar graph. By the outerplanarity lemma of [2], $G[V_{C'} \cap L_{d+1}]$ is outerplanar. We set $O := G[V_{C'} \cap L_{d+1}]$. The boundary curve(s) of $R_{C'}$ on the L_{d+1} side are exactly the boundary of O's outer face in the inherited embedding; this outer-face boundary is a single closed walk that traces around O from the outside, traversing any bridge edge twice and visiting cut-vertices multiple times. This walk is the inner boundary B_{in}. No further restriction on O's internal structure is needed: when the inner side has several lobes meeting through cut-vertices or bridges of O, the outer-face boundary closed walk of the connected graph O captures them by revisiting those vertices or traversing those bridges twice.

Tire structure. The triangular faces of $T_{C'}$ inside the closed boundary region are by construction the depth-d faces in $F_{C'}$, and the edges of $T_{C'}$ are $E(B_{\text{out}}) \cup E(O) \cup E_{\text{ann}}$ where E_{ann} are the edges of G between $V_{C'} \cap L_d$ and $V_{C'} \cap L_{d+1}$ that bound a face of $F_{C'}$. $\square$

Theorem 1.12 (Tire treads partition the bounded faces)**.** *Let G be a maximal planar graph with planar embedding Π_G and let $S \subseteq V(G)$ be a level source lying on the outer face. For each $d \geq 0$ and each connected component C' of G'_d, let $T^{(d,C')}$ denote the tire graph supplied by Lemma 1.11, with tire tread $R_{C'} \subseteq |\Pi_G|$. Then the collection of treads*

$$\mathcal{R}(G,S) \;:=\; \left\{\, R_{C'} \,:\, d \geq 0,\ C' \text{ a connected component of } G'_d \,\right\}$$

partitions the bounded part of $|\Pi_G|$:

(i) *every bounded face f of G is contained in exactly one tread $R_{C'} \in \mathcal{R}(G,S)$;*
(ii) *distinct treads in $\mathcal{R}(G,S)$ have disjoint interiors and may share only boundary edges or vertices.*

Proof. Existence and uniqueness. Each bounded face $f \in F(G)$ has a uniquely-defined dual depth $\delta_G(d_f) \in \mathbb{Z}_{\geq 0}$, so the dual vertex d_f lies in G'_d for $d = \delta_G(d_f)$ and in no other $G'_{d'}$. Within G'_d, the vertex d_f belongs to exactly one connected component C'. By Lemma 1.11, $F_{C'}$ is precisely the set of faces $f' \in F(G)$ with $d_{f'} \in V(C')$; in particular $f \in F_{C'}$, hence $f \subseteq R_{C'}$.

For any other tread $R_{C''} \in \mathcal{R}(G,S)$, the component C'' is either at a different depth $d' \neq d$ (in which case $F_{C''}$ consists of depth-d' faces and $f \notin F_{C''}$) or at depth d but a different component $C'' \neq C'$ (in which case the two components are vertex-disjoint in G'_d, so again $f \notin F_{C''}$). In both cases $f \notin R_{C''}$ (more precisely, f is not one of the triangular faces of G in $F_{C''}$, so f's interior is not contained in $R_{C''}$).

Disjoint interiors. Each tread $R_{C'}$ is the union of its triangular faces $F_{C'} \subseteq F(G)$; distinct treads correspond to disjoint $F_{C'}$ (by the argument above), and the interiors of distinct G-faces are disjoint. Hence interiors of distinct treads are disjoint.

Coverage. Conversely, every bounded $f \in F(G)$ has $d_f \in V(G')$ with some dual depth d, and thus lies in $R_{C'}$ where C' is its component of G'_d. So $\bigcup_{R \in \mathcal{R}(G,S)} R$ contains every bounded face of G. $\square$

Remark 1.13. Either boundary part of $T_{C'}$ in Lemma 1.11 may be degenerate. At $d = 0$ with single-vertex source $S = \{v_0\}$ the unique component of G'_0 has $V_{C'} \cap L_0 = \{v_0\}$ as the degenerate *outer* boundary and $V_{C'} \cap L_1$ a cycle (the link of v_0 in G) as the inner boundary. Symmetrically, at $d = D_{\max}$, $V_{C'} \cap L_{D_{\max}+1} = \emptyset$ degenerates to a single deepest vertex serving as the *inner* boundary, with the level-$D_{\max}$ cycle as the outer boundary.

Remark 1.14. Two structural features of $R_{C'}$ that might at first appear to obstruct the tire-graph conclusion are both already accommodated by Definition 1.6:

Cut-vertices of O. A vertex $v \in V_{C'} \cap L_{d+1}$ may have the faces of $F_{C'}$ incident to it split into two or more arcs in v's rotation in Π_G, separated by faces of higher depth. This corresponds exactly to v being a cut-vertex of $O = G[V_{C'} \cap L_{d+1}]$, and the inner boundary closed walk B_{in} then visits v multiple times — once for each arc. No additional hypothesis is needed.

Non-2-connected inner topology. Even when the inner side of $R_{C'}$ has several lobes joined at cut-vertices or across bridges, the connected inner outerplanar graph O captures this structure as a non-2-connected outerplanar graph, and its outer-face boundary closed walk serves as B_{in} by traversing bridges twice and visiting cut-vertices multiple times.

In the special case $d = 0$ with single-vertex source $S = \{v_0\}$, $R_{C'}$ is the star of v_0, a topological closed disk with one boundary cycle (the link of v_0); the corresponding tire graph has degenerate outer boundary $\{v_0\}$.

Theorem 1.15 (Inner dual of a tire tread is outerplanar)**.** *Let $T = (B_{\mathrm{out}}, O, E_{\mathrm{ann}})$ be a tire graph, and let Γ be the graph on vertex set $\{d_f : f \in F_{\mathrm{ann}}\}$ with an edge $d_f d_{f'}$ for each interior annular edge of T (= each edge of T whose two incident faces both lie in F_{ann}). Equivalently, Γ is the subgraph induced on F_{ann} of the* full tire dual *$D(T)$ — the dual of T taken over all of its triangular faces, in which each boundary edge of R contributes a degree-1 vertex. Then Γ is outerplanar.*

Moreover, Γ admits a planar embedding as a (possibly non-simple) Hamilton walk through every d_f, plus zero or more non-crossing chords.

Proof. We argue by cases on whether the tire tread R is a disk or an annulus.

Case 1: R is a closed disk (one of $B_{\mathrm{out}}, B_{\mathrm{in}}$ degenerate, by Definition 1.6). Let v_0 be the degenerate-boundary vertex (the apex) and let $k = |B_{\mathrm{non-deg}}|$ be the length of the non-degenerate boundary cycle. The triangulation of R is a *fan* of k triangles around v_0: each triangle has the form $\{v_0, u_i, u_{i+1}\}$ where $u_1, \ldots, u_k$ are the boundary-cycle vertices in cyclic order. Each triangle has two spoke edges (= the two edges incident to v_0, shared with the two neighbouring fan triangles) and one boundary edge (in $B_{\mathrm{non-deg}}$, contributing a leaf in $D(T)$ but no edge in Γ). Hence every d_f has Γ-degree exactly 2, and Γ is a single cycle of length k. Cycles are outerplanar.

See Figure 4 for the disk case ($k = 6$).

Case 2: R is an annulus (both B_{out} and B_{in} non-degenerate). We construct an explicit outerplanar embedding of Γ as a Hamilton walk plus non-crossing chords.

Step 1: Cyclic ordering of F_{ann}. The boundary of the annular tread is the disjoint union $\partial R = B_{\mathrm{out}} \sqcup \overline{B_{\mathrm{in}}}$ (viewing B_{in} as a closed walk traced in the appropriate orientation). Each boundary edge of R is incident to exactly one annular face: walking around B_{out} in cyclic order produces a sequence $f_1^{\mathrm{out}}, f_2^{\mathrm{out}}, \ldots, f_\mu^{\mathrm{out}}$ of (not necessarily distinct) annular faces, one per B_{out}-edge; similarly walking around B_{in} produces a sequence $f_1^{\mathrm{in}}, \ldots, f_{\nu_\partial}^{\mathrm{in}}$ where ν_∂ is the length of the inner-boundary walk. Pick any spoke $e^\star = uw \in E_{\mathrm{ann}}$ with $u \in V(B_{\mathrm{out}})$ and $w \in V(B_{\mathrm{in}})$; cut R along $e^\star$. This converts the annulus into a closed disk $\tilde{R}$ whose boundary walks once around B_{out}, once along $e^\star$, once around B_{in} in reverse, and once back along $e^\star$. Concatenating the two boundary sequences (in the order dictated by this disk traversal) yields a single cyclic sequence

$$\mathcal{S} = (f_1^{\mathrm{out}}, \ldots, f_\mu^{\mathrm{out}}, f_1^{\mathrm{in}}, \ldots, f_{\nu_\partial}^{\mathrm{in}})$$

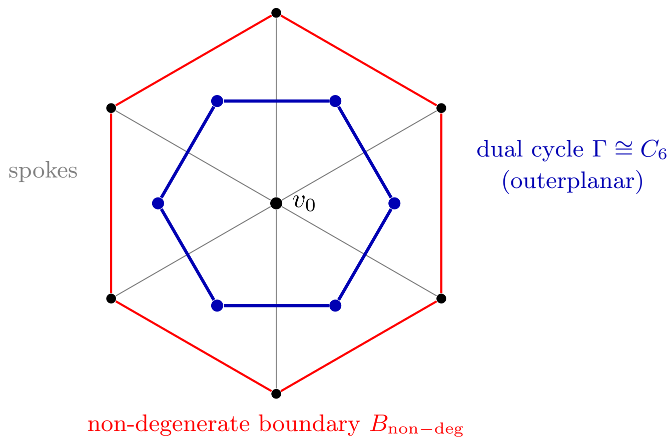

FIGURE 4. Case 1 (R = disk, $k = 6$). The apex v_0 sits at the centre; the non-degenerate boundary $B_{\text{non-deg}}$ (red) is the hexagonal outer cycle; spokes (grey) triangulate the disk into a fan of 6 triangles around v_0. Each triangle has two spoke edges (interior, contributing Γ-edges) and one boundary edge (contributing a leaf in $D(T)$, no Γ-edge). The inner dual Γ (blue) is the cycle C_6 formed by the six annular face centroids, a manifestly outerplanar graph.

of annular faces with multiplicities.

Step 2: The Hamilton walk. Consecutive entries of $\mathcal{S}$ correspond either to the same annular face (when two adjacent boundary edges meet at a vertex incident to a single annular face) or to two annular faces sharing an interior edge of E_{ann}. In the former case the walk stays at one Γ-vertex; in the latter it uses one Γ-edge. The resulting closed walk in Γ visits every face that appears in $\mathcal{S}$ at least once.

If every $f \in F_{\text{ann}}$ appears in $\mathcal{S}$ (i.e. every annular face has at least one boundary edge of R), the walk is a Hamilton walk in Γ, and we are done up to Step 3. Each annular face with two boundary edges contributes a vertex visited twice; each with three contributes a vertex visited three times.

If some $f \in F_{\text{ann}}$ does not appear in $\mathcal{S}$ (i.e. has no boundary edge of R), then all three edges of f are interior annular edges, so d_f has degree 3 in Γ. Such a face is "trapped" in the interior of the dual graph and appears as the endpoint of a chord. Extend the walk by: whenever it crosses an interior annular edge e shared with a boundary-free face f, detour through f and back. After finitely many such detours (one per boundary-free face), the walk becomes a Hamilton walk visiting every d_f.

Step 3: Non-crossing chords. The Γ-edges not used by the Hamilton walk constructed in Step 2 are the remaining interior annular edges. Each such edge $e \in E_{\text{ann}}$ corresponds to a chord between two non-adjacent positions of $\mathcal{S}$. In the inherited planar embedding of Γ in R, these chords are drawn as straight segments between annular triangle centroids; *they do not cross* because the underlying E_{ann} edges they cross are themselves non-crossing in the planar embedding of T.

Step 4: Outerplanar embedding. We now lay out Γ as follows: place the $|F_{\mathrm{ann}}|$ vertices on a circle in the cyclic order given by $\mathcal{S}$ (treating multiply-visited faces as single circle vertices). Connect consecutive vertices on the circle by the Hamilton-walk edges, which forms the closed walk. Draw the remaining edges as chords inside the circle. Because the chords were non-crossing in T's planar embedding, they remain non-crossing here. All vertices lie on the outer face (the unbounded region outside the circle), making Γ outerplanar. □ □

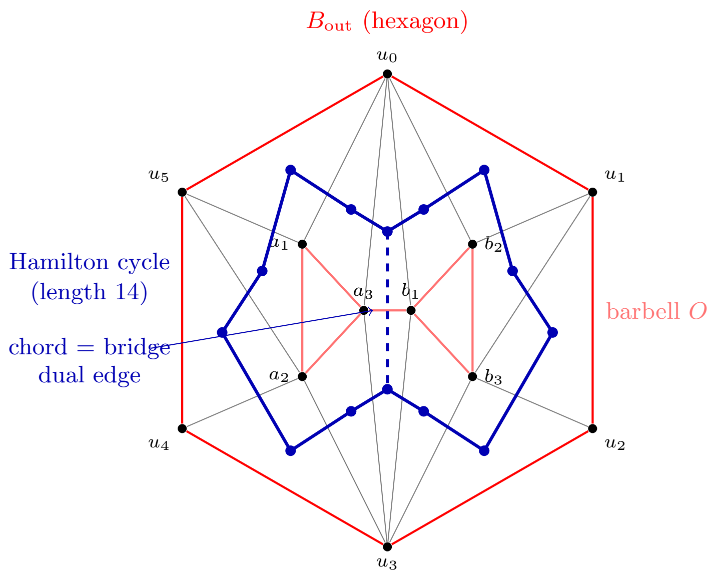

FIGURE 5. Case 2 (R = annulus) with O a barbell. B_{out} is the outer hexagon (red); O has two triangles $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$ joined by the bridge a_3–b_1 (all light red). The annulus is triangulated by 14 annular triangles: 6 "outer-cap" triangles (one per outer edge), 6 "inner-cap" triangles (one per non-bridge edge of O), and 2 "bridge-cap" triangles $\{u_0, a_3, b_1\}$ and $\{u_3, a_3, b_1\}$ adjacent to the bridge. Each blue dot sits at the centroid of an annular triangle; blue edges connect dual vertices whose triangles share an interior annular edge (spoke or bridge). The two bridge-cap vertices have Γ-degree 3 (their triangles have no boundary edge) and are joined by the dashed blue *chord* corresponding to the bridge; the remaining 13 edges form the Hamilton cycle that wraps around the annulus. All 14 vertices lie on the outer face of the cycle-with-chord embedding, so $\Gamma \cong \Theta(1, 7, 7)$ is outerplanar.

Remark 1.16. In the *spoke-only* case (Definition 1.6 with O 2-connected and E_{ann} consisting only of spokes), every annular face has exactly one boundary edge, every d_f has Γ-degree 2, and the construction of the Theorem 1.15 proof reduces to the classical Hamilton cycle $\Gamma \cong C_{\mu+\nu}$ with zero chords.

Remark 1.17. When O has a bridge $e_{\mathrm{br}} \in E(O)$ whose two incident faces are annular triangles, e_{br} contributes an interior annular edge in Γ rather than two leaves in $D(T)$ (see Definition 1.7 of [3]). The two bridge-incident annular triangles have Γ-degree 3; the resulting Γ has the structure of a Hamilton cycle of length $\mu + \nu_\partial$ plus a single chord (length 1). This corresponds to the theta graph $\Theta(1, b, c)$ identified empirically in [3], which has no $K_{2,3}$ subdivision (since one of the three paths has length 1 and so contributes no degree-2 branch vertex), hence is outerplanar as predicted.

Theorem 1.18 (Tait correspondence: 4-colorings of a tire vs 3-edge-colorings of its inner dual)**.** *Let $T = (B_{\mathrm{out}}, O, E_{\mathrm{ann}})$ be a tire graph (viewed as an annular triangulation of its tire tread R) and let Γ be its inner dual (Theorem 1.15). Then*

$$\#\{\textit{proper 4-vertex-colorings of } T\}/|S_4| \;=\; \#\{\textit{proper 3-edge-colorings of } \Gamma\}/|S_3|.$$

That is, the number of 4-vertex-colorings of T up to permutation of the colour set $\{0, 1, 2, 3\}$ equals the number of 3-edge-colorings of Γ up to permutation of the colour set $\{1, 2, 3\}$.

Proof. The argument is the classical Tait correspondence [1] adapted to the annular triangulation T. Encode the four colours of a proper 4-vertex-coloring $c\colon V(T) \to \mathbb{Z}_2 \times \mathbb{Z}_2$. For each interior annular edge e of T (whose two incident faces both lie in F_{ann}, contributing a Γ-edge e^*), set

$$\chi^*(e^*) \;:=\; c(u) + c(v) \in \mathbb{Z}_2 \times \mathbb{Z}_2, \qquad \text{where } u, v \text{ are the endpoints of } e.$$

Since $c(u) \neq c(v)$, we have $\chi^*(e^*) \neq 00$, so χ^* takes values in $\{01, 10, 11\}$, which we identify with the 3-edge-coloring palette $\{1, 2, 3\}$.

Properness. At each Γ-vertex d_f corresponding to an annular triangle $f = \{u, v, w\}$, the three incident Γ-edges (one per cycle-edge of f) carry colours $c(u) + c(v)$, $c(v) + c(w)$, $c(u) + c(w)$. These three elements of $\mathbb{Z}_2 \times \mathbb{Z}_2$ sum to 0 and are pairwise distinct (their pairwise differences are $c(u) - c(w)$, $c(v) - c(u)$, $c(w) - c(v)$, each nonzero), so they form a permutation of $\{01, 10, 11\}$ — a proper edge colouring at d_f.

Surjectivity onto cosets. Given a proper 3-edge-coloring χ^* of Γ, the equation $c(u) + c(v) = \chi^*(e^*)$ admits exactly $|\mathbb{Z}_2 \times \mathbb{Z}_2| = 4$ solutions $c\colon V(T) \to \mathbb{Z}_2 \times \mathbb{Z}_2$ (a global translation is the only freedom). Hence the map $c \mapsto \chi^*$ is 4-to-1.

Count. Therefore $\#\{\text{4-colorings of } T\} = 4 \cdot \#\{\text{3-edge-colorings of } \Gamma\}$. Dividing by $|S_4| = 24$ on the left and $|S_3| = 6$ on the right (since S_4 acts faithfully on the 4-colorings and S_3 on the 3-edge-colorings, and the 4-to-1 map respects the $S_4/S_3 \cong S_3$ quotient via the natural surjection $S_4 \twoheadrightarrow S_3$) gives the stated equality. ☐

Remark 1.19. Theorem 1.18 reduces the 4-colouring count of a tire to the 3-edge-coloring count of its outerplanar inner dual Γ. For the cycle case $\Gamma \cong C_{\mu+\nu}$ (the spoke-only case of Remark 1.16), the cycle chromatic polynomial at 3 colours gives $2^{\mu+\nu} + 2(-1)^{\mu+\nu}$. For an inner dual with one or more non-crossing chords, the count depends on the chord structure, not just on the pair (number of vertices, number of chords): two outerplanar graphs with the same number of vertices and number of chords can have different proper 3-edge-coloring counts depending on how the chords are arranged (nested, sequential, sharing vertices, etc.). Every such count can nevertheless be computed in linear time by tree-decomposition methods, since outerplanar graphs have treewidth at most 2 and the edge-chromatic polynomial admits a deletion–contraction recursion that respects the cycle-plus-chord structure.

Definition 1.20 (Boundary-state chromatic transfer)**.** Let $T = (B_{\mathrm{out}}, O, E_{\mathrm{ann}})$ be a tire graph. Choose a cut along one annular edge if both boundaries are non-degenerate; in the degenerate case make no cut. The tread becomes a triangulated disk $\widetilde{R}$. Let

$$f_1, f_2, \dots, f_m$$

be any shelling order of the triangular faces of $\widetilde{R}$, i.e. an order in which each initial union $\widetilde{R}_i := f_1 \cup \cdots \cup f_i$ is a disk. Such an order is obtained by taking an outerplanar embedding of the inner dual Γ from Theorem 1.15 and repeatedly removing an outer-face ear.

For each i, let A_i be the *frontier*: the vertices of T incident to at least one processed face in $\widetilde{R}_i$ and to at least one still-unprocessed constraint, where the unprocessed constraints are the remaining annular faces together with any edge of O not yet tested by the transfer. A *boundary state* on A_i is a partition π of A_i into colour classes, subject to the condition that adjacent vertices of $T[A_i]$ lie in distinct blocks. We write $r(\pi)$ for the number of blocks of π.

Theorem 1.21 (Chromatic polynomial of a tire by frontier transfer)**.** *For every tire graph T, the chromatic polynomial $P_T(q)$ is computed by the following boundary-state dynamic program.*

Initialize the table at $i = 0$ with the empty frontier state of weight 1. When the next triangular face $f_i = \{x, y, z\}$ is attached, pass from states on A_{i-1} to states on A_i as follows.

1. *Introduce any vertices of f_i not already present in A_{i-1}, assigning each such vertex either to an existing colour block not containing one of its already-coloured neighbours, or to a new block.*
2. *Reject every assignment in which two adjacent vertices of the triangle f_i lie in the same block. Also reject every assignment in which an edge of O whose two endpoints have now both appeared for the first time as a tested pair has both endpoints in the same block. Thus chords and bridges of the inner outerplanar graph are enforced exactly when their second endpoint becomes visible to the transfer.*
3. *Delete from the state every vertex no longer incident to an unprocessed constraint. If deleting a vertex removes the last representative of its colour block from the frontier, multiply that transition by 1; the colour has already been chosen.*
4. *If a new vertex is assigned to a new colour block while the current frontier state has r colour blocks, multiply that transition by $q - r$. If several new colour blocks are created in the same triangle, the factors are $(q - r)(q - r - 1)\cdots$ in the order of creation.*

After f_m is processed, the frontier is empty. The single remaining weight is $P_T(q)$.

Proof. The construction is the standard transfer for the chromatic polynomial, specialized to the tire shelling. The frontier state records exactly the equality pattern among colours that can still affect unprocessed faces. Since colour names are irrelevant to the chromatic polynomial, states are quotiented by the natural action of the symmetric group on the colour set; a state with r visible colour blocks can be extended by a genuinely new colour in $q - r$ ways.

Each transition accounts for all proper colourings of the enlarged processed disk $\widetilde{R}_i$ that restrict to the resulting frontier state, and accounts for none that violate an edge of the newly attached triangle or untested edge of O. Vertices removed from the frontier have no future incident unprocessed constraint, so their actual colour names can no longer influence compatibility and may be forgotten. Induction on i therefore shows that the table after step i is precisely the orbit-count generating function for proper colourings of $\widetilde{R}_i$ by frontier state. At $i = m$ no vertices remain active, so the accumulated weight counts all proper colourings of T. Because the weights are polynomials in q, this count is the full chromatic polynomial. □

Remark 1.22 (Spoke-only transfer matrix). In the spoke-only case with both boundaries simple cycles, the method has a particularly small form. Cut the annulus along one spoke and walk around the resulting strip. Each step adds one triangle sharing an edge with the previous processed strip, so the frontier consists of two or three consecutive boundary vertices. Up to colour permutation there are only the possible equality patterns among those active vertices, with adjacent vertices required to be distinct. The chromatic polynomial is therefore the trace of a finite transfer matrix whose entries are polynomials in q; the matrix depends only on the local triangle type encountered while walking around the tread. Chords or cut-vertices of O enlarge the frontier only at the corresponding outerplanar ears, and are handled by the same state rule of Theorem 1.21.

Theorem 1.23 (Tire treads form a rooted tree under face containment)**.** *Let G be a maximal planar graph with planar embedding Π_G and let $S \subseteq V(G)$ be a single-vertex level source $\{v_0\}$ lying on the outer face of Π_G. The collection $\mathcal{R}(G,S)$ of tire treads (Theorem 1.12) carries a canonical rooted tree structure $\mathcal{T}(G,S)$ defined as follows.*

- Root. *The depth-*0 *tire tread T_0 — the unique tire produced by Lemma 1.11 at $d = 0$, with degenerate outer boundary $B_{\text{out}} = \{v_0\}$ and inner outerplanar graph $O^{(T_0)} = G[L_1]$ — is the root.*
- Parent. *For each tire tread T_c at depth $d \geq 1$, its outer boundary $B_{\text{out}}^{(T_c)}$ is a cycle in L_d. The* parent *of T_c is the unique tire tread T_p at depth $d-1$ whose inner outerplanar graph $O^{(T_p)}$ has $B_{\text{out}}^{(T_c)}$ as the boundary cycle of one of its bounded faces. Equivalently, R_c lies inside this bounded face of $O^{(T_p)}$ (which is itself the region of the plane cut off by $B_{\text{out}}^{(T_c)}$ on the side away from S).*
- Children. *The children of a tire tread T_p are in bijection with those bounded faces of $O^{(T_p)}$ whose interiors contain at least one vertex of G at level $\geq d+2$ — equivalently, with the connected components of G'_{d+1} whose tires have outer boundary cycle equal to a bounded face of $O^{(T_p)}$.*

Every tire tread except T_0 has exactly one parent; a tire tread may have zero, one, or several children.

Proof. Root is well-defined. At $d = 0$ with single-vertex source $S = \{v_0\}$, the dual subgraph G'_0 is connected (every face of G incident to v_0 has dual depth 0, and they form a single fan around v_0). By Lemma 1.11, the unique component of G'_0 gives the depth-0 tire T_0 described above.

Existence of parent. Fix a tire tread T_c at depth $d \geq 1$ arising from a connected component C'_c of G'_d. Its outer boundary $B_{\text{out}}^{(T_c)} = G[V_{C'_c} \cap L_d]$ is a simple cycle in L_d (Lemma 1.11; the source-side boundary of a tire is always a simple cycle, by Proposition 1.10). The faces of G immediately outside $B_{\text{out}}^{(T_c)}$ on the side facing S

have depth $d-1$ (one of their three vertices lies in L_{d-1}, two in L_d). Let C'_p be the connected component of G'_{d-1} containing the dual vertex of any such face.

Uniqueness of parent. $B^{(T_c)}_{\text{out}}$ is a single simple cycle in G, with a well-defined "S-side" (the side of the cycle closer to v_0 in Π_G). The depth-$(d-1)$ faces lying on this side form a single contiguous arc around $B^{(T_c)}_{\text{out}}$ in the dual — they are all G'-adjacent in sequence (each pair of consecutive arc faces shares an edge in $B^{(T_c)}_{\text{out}}$). Hence they all lie in the same connected component C'_p of G'_{d-1}, which is therefore unique.

$B^{(T_c)}_{\text{out}}$ *bounds a face of* $O^{(T_p)}$. The parent tire T_p has $V(O^{(T_p)}) = V_{C'_p} \cap L_d \supseteq V(B^{(T_c)}_{\text{out}})$. The cycle $B^{(T_c)}_{\text{out}}$ is a subgraph of $O^{(T_p)}$ that bounds a face of $O^{(T_p)}$ in the inherited embedding: the cycle traces around a depth-$\geq d+1$ region (containing R_c and any descendants of T_c), which is exactly a bounded face of $O^{(T_p)}$.

Children description. The bounded faces of $O^{(T_p)}$ are in bijection with the connected components of G'_d whose faces lie inside those bounded regions (= one component per bounded face, by an argument analogous to the existence-and- uniqueness step above, applied one level deeper).

Tree property. Every non-root T_c has a unique parent at strictly smaller depth. Iterating the parent map strictly decreases depth, terminating at T_0. No cycles can form (depth is monotone). Hence $\mathcal{T}(G,S)$ is a rooted tree. □

Remark 1.24. A parent tire T_p has multiple children precisely when its inner outerplanar graph $O^{(T_p)}$ has multiple bounded faces with non-trivial interiors (= containing depth-$\geq d+2$ vertices of G). This happens, for instance, when $O^{(T_p)}$ has chords or cut-vertices that subdivide its inner region. By contrast, if $O^{(T_p)}$ is a simple cycle (the spoke-only case of Remark 1.16) with a non-empty interior, T_p has exactly one child.

Theorem 1.25 (Tire-tree decomposition)**.** *Let G be a maximal planar graph with planar embedding Π_G and let $v_0 \in V(G)$. The tree of tire treads $\mathcal{T}(G,\{v_0\})$ of Theorem 1.23* decomposes G into nested tires*: it is a finite rooted tree, rooted at the depth-*0 *tread containing v_0, whose nodes (tire treads) partition the bounded faces of G (Theorem 1.12).*

This decomposition is moreover self-similar*. For any tread T in $\mathcal{T}(G,\{v_0\})$ at depth $d \geq 1$, with outer-boundary cycle $C_T := B^{(T)}_{\text{out}}$, let G_T be the sub-graph of G induced by C_T together with all vertices of G lying in the closed planar region $R_T \subset |\Pi_G|$ bounded by C_T on the side of C_T away from v_0. Then:*

- (D1) *G_T, with the embedding inherited from Π_G, is a* triangulated disk*: every bounded face is a triangle, and the outer face is bounded by C_T.*
- (D2) *Taking C_T as a cycle source of G_T (so C_T has level* 0 *in G_T and the BFS-from-C_T levels in G_T equal $\ell_G(\cdot) - d$ on $V(G_T)$), the construction of Theorem 1.23 extends to give a rooted tree of tire treads $\mathcal{T}(G_T, C_T)$ whose depth-*0 *root tread has $B_{\text{out}} = C_T$ and inner outerplanar graph $O = O^{(T)}$.*
- (D3) *$\mathcal{T}(G_T, C_T)$ is canonically iso to the sub-tree of $\mathcal{T}(G,\{v_0\})$ rooted at T, preserving outer-boundary cycles, inner outerplanar graphs, and the parent–child face correspondence.*

In short: pick any vertex $v_0 \in V(G)$ to root the global tree $\mathcal{T}(G,\{v_0\})$ describing the whole graph; pick any tread T in this tree; then T is itself the root of a local *tree $\mathcal{T}(G_T, C_T)$ describing the triangulated disk of G inside C_T, with C_T as cycle source. Maximal planar graphs decompose into nested trees of tire treads.*

Proof. Decomposition. Theorem 1.23 gives the rooted tree structure of $\mathcal{T}(G, \{v_0\})$, with root the depth-0 tread containing v_0; Theorem 1.12 gives that its tire treads partition the bounded faces of G. Finiteness of the tree is immediate from finiteness of G.

(D1) G_T is a triangulated disk. By Lemma 1.11 applied to the component of G'_d that gives rise to T, the outer boundary $C_T = B^{(T)}_{\text{out}}$ is a simple cycle in L^G_d. By the Jordan curve theorem, C_T separates $|\Pi_G| \setminus C_T$ into two open regions; R_T is the closure of the one not containing v_0. The bounded faces of G_T in its inherited embedding are exactly the bounded faces of G contained in R_T, each of which is a triangle since G is a triangulation. The unbounded face of G_T's embedding is the complement of R_T, whose boundary is C_T.

(D2) Level shift. We show $\operatorname{dist}_{G_T}(v, C_T) = \ell_G(v) - d$ for every $v \in V(G_T)$. When $v \in C_T$ both sides equal 0, so fix $v \in V(G_T) \setminus C_T$.

Step 1: $\operatorname{dist}_G(v, C_T) = \ell_G(v) - d$. A shortest G-path from v to v_0 must visit C_T, since v and v_0 lie in different open regions of $|\Pi_G| \setminus C_T$; let w be its first C_T-vertex. The v-to-w sub-path has length $\geq \operatorname{dist}_G(v, C_T)$ and the w-to-v_0 sub-path has length $\ell_G(w) = d$, so $\ell_G(v) \geq \operatorname{dist}_G(v, C_T) + d$. Conversely, concatenating a shortest G-path from v to a nearest C_T-vertex w' with a shortest G-path from w' to v_0 gives a v-to-v_0 path of length $\operatorname{dist}_G(v, C_T) + d$, so $\ell_G(v) \leq \operatorname{dist}_G(v, C_T) + d$.

Step 2: $\operatorname{dist}_{G_T}(v, C_T) = \operatorname{dist}_G(v, C_T)$. The inequality $\geq$ is automatic since $G_T \subseteq G$. For $\leq$, pick a shortest G-path π from v to C_T; we may assume π has no internal vertex in C_T (truncate otherwise). Any internal vertex of π then lies in the same open region of $|\Pi_G| \setminus C_T$ as v, i.e. in $R_T \setminus C_T \subseteq V(G_T)$; every edge of π has both endpoints in $V(G_T)$ and so lies in $E(G_T)$. Hence π is a path in G_T realising $\operatorname{dist}_G(v, C_T)$.

Combining the two steps yields $\operatorname{dist}_{G_T}(v, C_T) = \ell_G(v) - d$, as claimed.

(D3) Tree iso. By (D2), $L^{G_T}_k = L^G_{d+k} \cap V(G_T)$ for every $k \geq 0$. For a bounded face f of G_T, dual depth in G_T equals $\min_{u \in V(f)} \ell_{G_T}(u) = \min_{u \in V(f)} \ell_G(u) - d = \delta_G(d_f) - d$. Hence the inner-dual subgraph $(G_T)'_k$ at depth k in G_T is the induced subgraph of G'_{d+k} on the faces of G lying in R_T, and two such faces are dual-adjacent in G'_T iff they are dual-adjacent in G' (the shared edge is in $E(G_T)$).

Step 3: components of $(G_T)'_k$ are precisely the depth-$(d+k)$ descendants of T in $\mathcal{T}(G, \{v_0\})$. We show by induction on k that a component C' of G'_{d+k} has $F_{C'} \subseteq R_T$ iff C' is a depth-$(d+k)$ descendant of T.

For $k = 0$: the components of G'_d are the depth-d treads; the component giving rise to T has its faces in T's tread region $R \subseteq R_T$, while any other depth-d tread T'' has $C_{T''}$ disjoint from C_T and lying in a different bounded face of $O^{(T''_p)}$ at depth $d-1$, hence $R_{T''} \cap R_T = \emptyset$.

For $k \geq 1$: by Theorem 1.23, each component C' of G'_{d+k} has a unique parent C'_p at depth $d+k-1$, with $B^{(C')}_{\text{out}}$ bounding a face of $O^{(C'_p)}$; equivalently $R_{C'}$ lies inside that bounded face, hence inside $R_{C'_p}$. By the induction hypothesis $R_{C'_p} \subseteq R_T$ iff C'_p is a descendant of T at depth $d+k-1$, and $R_{C'} \subseteq R_{C'_p}$, so $R_{C'} \subseteq R_T$ iff C' is a descendant of T at depth $d+k$.

Step 4: tread data and child–face correspondence. The Tire-component lemma (Lemma 1.11) and the source-side simple-cycle property (Proposition 1.10) extend verbatim to the cycle-sourced triangulated disk (G_T, C_T): the proofs use only the triangular structure of bounded faces, the local arrangement of faces around each vertex's rotation, and the connectivity of the BFS ball $G_T[L^{G_T}_{<k}]$ (which holds for every $k \geq 1$ since $L^{G_T}_0 = V(C_T)$ is connected as a cycle and each higher level is BFS-adjacent to the previous). Applied to each component of $(G_T)'_k$, the lemma produces a tire graph with outer boundary B_{out}, inner outerplanar graph O, and tread region R identical to those produced by the corresponding component of G'_{d+k} in $\mathcal{T}(G, \{v_0\})$, since these data depend only on level-$d{+}k$ and level-$(d{+}k{+}1)$ vertices and the bounded faces in between — all of which are unchanged when restricting to G_T.

The depth-0 case ($k = 0$) gives a single component, namely the one producing T, with root tread $B_{\text{out}} = C_T$ and $O = O^{(T)}$.

The parent–child face correspondence of Theorem 1.23 is preserved: for any tread T' in $\mathcal{T}(G_T, C_T)$ at depth k, its children correspond to non-trivial bounded faces of $O^{(T')}$, and the bounded faces of $O^{(T')}$ together with the descendant-side interior of each are identical in G_T and in G.

Combining Steps 3 and 4: the bijection $C' \leftrightarrow C'$ (component of $(G_T)'_k$ to corresponding component of G'_{d+k} inside R_T) lifts to a rooted-tree iso $\mathcal{T}(G_T, C_T) \to$ sub-tree of $\mathcal{T}(G, \{v_0\})$ rooted at T, preserving outer boundaries, inner outerplanar graphs, and the parent–child face correspondence. $\square$

Remark 1.26. Combining Theorem 1.12 (treads partition the bounded faces of G) with Theorem 1.23 (treads form a rooted tree), any proper coloring problem on G's bounded faces factors through:

- local coloring problems on each tread (the inner dual of each tread is outerplanar by Theorem 1.15), plus
- consistency constraints along parent-child interfaces (the cycle $B^{(T_c)}_{\text{out}}$ shared between a child and the face of its parent's $O^{(T_p)}$).

This is the structural setup underlying the chain-pigeonhole program for tire treads.

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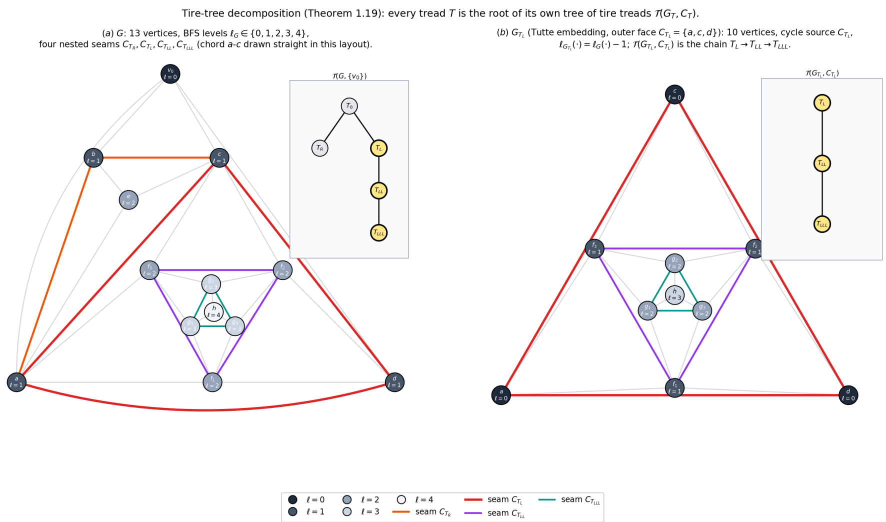

FIGURE 6. Tire-tree decomposition (Theorem 1.25) on a 13-vertex maximal planar example G with five BFS levels. (a) G with vertex source v_0 and $\ell_G \in \{0,1,2,3,4\}$; four nested seams are highlighted, $C_{T_R} = \{a,b,c\}$ (orange), $C_{T_L} = \{a,c,d\}$ (red, including the chord a-c shared with C_{T_R}), $C_{T_{LL}} = \{f_1, f_2, f_3\}$ (purple), $C_{T_{LLL}} = \{g_1, g_2, g_3\}$ (teal). Inset: the rooted tree of tire treads $\mathcal{T}(G, \{v_0\})$ branches at T_0 into the leaf T_R (containing e) and a chain $T_L \to T_{LL} \to T_{LLL}$ (the highlighted sub-tree). (b) The disk G_{T_L} inside the seam C_{T_L}, drawn standalone with C_{T_L} as cycle source and vertex labels rotated to match the new (cycle-source) role of the boundary triangle. $\ell_{G_{T_L}}(\cdot) = \ell_G(\cdot) - 1$ on $V(G_{T_L})$ (verified by the generator script), and $\mathcal{T}(G_{T_L}, C_{T_L})$ is the chain $T_L \to T_{LL} \to T_{LLL}$, iso to the highlighted sub-tree of (a).