COLORING NESTED TIRE GRAPHS

ERIC BAUERFELD

Abstract.

1. Introduction

A classical theorem of Tait recasts the Four Colour Theorem in dual, edge-colouring terms: a plane triangulation G is properly 4-vertex-colourable if and only if its dual cubic graph G' is properly 3-edge-colourable. Thus a minimal counterexample to the Four Colour Theorem – a smallest triangulation admitting no proper 4-colouring – corresponds to a smallest cubic plane graph admitting no proper 3-edge-colouring.

We study the structure such a minimal counterexample would have to exhibit through the lens of *nested level duals*. Fixing a level source S in G endows the dual G' with a Breadth-First-Search–derived labelling, the dual depth of Definition 1.4, and the level structure of G organises G' into a family of nested cycles carrying these labels. Our aim is to express the obstruction to a 3-edge-colouring of G' as conditions on this nested labelled-cycle structure.

Throughout, $G = (V, E)$ is a plane maximal planar graph (a triangulation) with a fixed planar embedding Π_G. We write $|V| = n$, so $|E| = 3n - 6$ and G has $2n - 4$ triangular faces.

Definition 1.1 (Level source)**.** A *level source* of G is any vertex $v \in V$; we write $S = \{v\}$ for the level-0 source.

Definition 1.2 (Levels)**.** Given a level source $S \subseteq V$, the *level* of $v \in V$ is $\ell_G(v) = \mathrm{dist}_G(v, S)$, the graph distance from v to the nearest source vertex.

Definition 1.3 (Dual)**.** The *dual* of G, written G', is the inner (weak) planar dual of G with respect to the embedding Π_G: it has one vertex d_f for each bounded face f of G, and an edge joining d_f and $d_{f'}$ for each edge of G shared by two bounded faces f and f'. The unbounded outer face contributes no vertex, and edges of G on the outer boundary contribute no dual edge. Since G is a triangulation, each vertex $d_f \in V(G')$ corresponds to a triangular face f of G, and we write $V(f) \subseteq V$ for its three incident vertices.

Definition 1.4 (Dual depth)**.** Given a level source $S \subseteq V$, the *dual depth* of a dual vertex $d_f \in V(G')$ is

$$\delta_G(d_f) = \min_{v \in V(f)} \ell_G(v) = \min_{v \in V(f)} \mathrm{dist}_G(v, S),$$

the smallest level among the three vertices of G bounding the face f.

2010 *Mathematics Subject Classification.* Primary .

Key words and phrases. plane graph, triangulation, plane depth, level edge, dual graph.

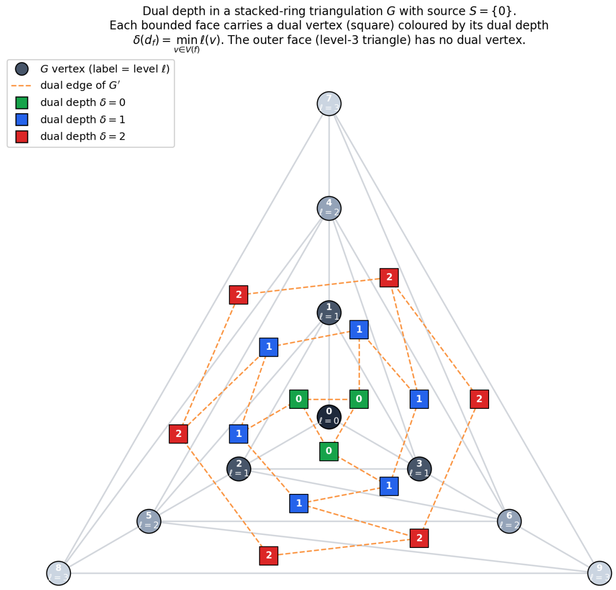

FIGURE 1. Dual depth in a stacked-ring triangulation G with level source $S = \{0\}$. Each G vertex is labelled by its level ℓ. Each bounded face carries a dual vertex (square, joined by dashed dual edges) coloured by its dual depth $\delta(d_f) = \min_{v \in V(f)} \ell(v)$: the central fan has depth 0, the inner annulus depth 1, and the outer annulus depth 2. The outer face (the level-3 triangle) is excluded from the inner dual and carries no dual vertex.

Definition 1.5 (Tire graph)**.** Let C_{out} be a simple cycle of length $m \geq 3$, and let O be an outerplanar graph whose outer-face boundary C_{in} is a simple cycle of length $k \geq 3$, with $V(C_{\text{out}}) \cap V(O) = \emptyset$. A *tire graph* on (C_{out}, O) is a plane graph T with

$$V(T) = V(C_{\text{out}}) \cup V(O), \qquad E(T) = E(C_{\text{out}}) \cup E(O) \cup E_{\text{ann}},$$

where E_{ann} is a set of edges — the *annular edges* — such that, in the plane embedding of T, the closed annulus with outer boundary C_{out} and inner boundary C_{in} is partitioned into triangular faces. Equivalently, the bounded faces of T that are not faces of O are all triangles, and together they tile the annular region between C_{out} and C_{in}.

We call C_{out} the *outer cycle*, O the *inner outerplanar graph*, and C_{in} the *inner cycle* of T. When $O = C_{\text{in}}$ (the inner outerplanar graph has no chords), T is a tire graph *with empty inner*; in general O contributes only chords inside the disk bounded by C_{in} and does not interact with E_{ann}.

Remark 1.6. A tire graph on (C_{out}, O) has $|V(C_{\text{out}})| + |V(O)| = m + k$ vertices, exactly $m + k$ annular triangles in the annulus between C_{out} and C_{in} (by Euler's formula on the annulus), and exactly $m + k$ annular edges in E_{ann}, of which the $m + k$ triangles share their three edges with the boundaries $E(C_{\text{out}}) \cup E(C_{\text{in}})$ and with each other.