

DUAL DECOMPOSITION OF MINIMAL COUNTEREXAMPLES

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ABSTRACT.

1. THE MINIMAL COUNTEREXAMPLE

Throughout, a *triangulation* is a simple plane graph, with a fixed embedding, in which every face — including the outer face — is bounded by a triangle. We first reduce to triangulations, then record the degree properties a smallest counterexample must have.

Lemma 1.1 (Reduction to triangulations). *If every triangulation is properly 4-vertex-colourable, then so is every plane graph.*

Proof. Let H be a plane graph. Add edges to H , maintaining planarity, until no further edge can be added; the result is a triangulation H^+ on the same vertex set with $E(H) \subseteq E(H^+)$. A proper 4-colouring of H^+ restricts to a proper 4-colouring of H , since every edge of H is an edge of H^+ . \square

By Lemma 1.1, if the Four Colour Theorem fails then it fails for some triangulation. We may therefore make the following assumption.

Definition 1.2 (Minimal counterexample). Let G be a triangulation on the fewest vertices that admits no proper 4-vertex-colouring. We call G a *minimal counterexample*. By minimality, every triangulation on fewer than $|V(G)|$ vertices is properly 4-colourable.

Remark 1.3. Since every triangulation on at most four vertices is properly 4-colourable (the largest being K_4), a minimal counterexample has $|V(G)| \geq 5$; the degree bound below sharpens this to $|V(G)| \geq 12$.

Lemma 1.4 (Minimum degree). *A minimal counterexample G has minimum degree $\delta(G) \geq 5$.*

Proof. Suppose some vertex v has $\deg(v) = d \leq 4$.

If $d \leq 3$, let $G' = G - v$. Then G' is a plane graph on fewer vertices, so by Definition 1.2 and Lemma 1.1 it has a proper 4-colouring. The at most three neighbours of v use at most three colours, so a fourth colour is free for v , extending the colouring to G — a contradiction.

If $d = 4$, again 4-colour $G - v$. If the four neighbours of v use at most three colours we extend as before, so assume they receive all four colours; let v_1, v_2, v_3, v_4 be the neighbours in cyclic order around v , coloured 1, 2, 3, 4. Consider the subgraph

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induced by the colour classes 1 and 3, and let K be its connected component containing v_1 . If $v_3 \notin K$, swap colours 1 and 3 on K ; now no neighbour of v is coloured 1, freeing it for v . If $v_3 \in K$, then a 1–3 Kempe chain joins v_1 to v_3 , and this chain together with v encloses exactly one of v_2, v_4 ; hence the 2–4 component containing v_2 cannot also reach v_4 , and swapping colours 2 and 4 on it frees colour 2 for v . Either way the colouring extends to G , a contradiction.

Hence $\delta(G) \geq 5$. □