

Level Resolutions of Maximal Planar Graphs: A Proof Strategy for the Four Color Theorem and Computational Investigation of Surjectivity

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(with computational verification by Claude)

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Abstract

We propose a structural reformulation of the four color theorem in terms of *level resolutions* of maximal planar graphs. A level structure on a plane graph G is defined by BFS from a chosen level source (either a face or a degree-3 vertex), partitioning vertices into levels. A triangulation G' on the same vertex set is a *level resolution* of G from this source if the subgraphs of G' induced by even- and odd-level vertices are both bipartite. By construction, any level resolution admits an explicit 4-coloring obtained by 2-coloring each parity subgraph independently. The structural foundation of this approach is that each level subgraph L_k of G is outerplanar (verified for all triangulations and sources at $n \leq 10$), and outerplanar graphs are 3-chromatic; the level-resolution problem is precisely to flip edges of G to reduce each L_k from chromatic number 3 to 2. We present computational results characterizing which isomorphism classes of maximal planar graphs on $n = 6, \dots, 11$ vertices arise as level resolutions, and verify that every iso-class is reachable at every tested size.

1 Introduction

The four color theorem (4CT) asserts that every planar graph is 4-colorable. Equivalently, every maximal planar graph (triangulation) is 4-colorable. The Appel–Haken proof [1] and subsequent Robertson–Sanders–Seymour–Thomas refinement [2] rely on discharging arguments and computer-verified reducible configurations. Human-readable proofs remain elusive.

We propose a different structural approach. Given a plane triangulation G and a choice of *level source*, BFS from the source partitions the vertices into levels. A triangulation G' on the same vertex set is a *level resolution* of G if, when its vertices are labelled by the parity of their G -levels, the subgraph of G' induced by even-parity vertices and the subgraph induced by odd-parity vertices are both bipartite. The 4-coloring of G' then follows by definition: 2-color each parity subgraph and identify the four resulting classes with four distinct colors.

The remaining question is when level resolutions exist. We conjecture:

- (i) every plane triangulation G' is a level resolution of some plane triangulation G via some level source; or, in a restricted form,
- (ii) every plane triangulation of minimum degree at least 4 is a level resolution of some plane triangulation.

This paper formalizes the definitions and presents computational evidence bearing on (i)–(ii) for small vertex counts.

2 Definitions

Throughout, $G = (V, E)$ is a plane maximal planar graph (a triangulation) with a fixed planar embedding Π_G . We write $|V| = n$, so $|E| = 3n - 6$ and G has $2n - 4$ triangular faces.

Definition 1 (Level source). A *level source* of G is either:

- a face F of G (all vertices of F are level-0 sources), or
- a vertex v of degree 3 (the singleton $\{v\}$ is a level-0 source).

Definition 2 (Levels). Given a level source $S \subseteq V$, the *level* of $v \in V$ is $\ell_G(v) = \text{dist}_G(v, S)$, the graph distance from v to the nearest source vertex.

Definition 3 (Parity subgraph). Let G be a triangulation with level source S , and let G' be a triangulation on the same vertex set as G . The *even parity subgraph* $E_{G,S}(G')$ is the subgraph of G' induced by $\{v \in V : \ell_G(v) \equiv 0 \pmod{2}\}$. The *odd parity subgraph* is defined analogously for odd ℓ_G .

Definition 4 (Level resolution). A triangulation G' on the same vertex set as G is a *level resolution* of G from level source S if both the even and odd parity subgraphs $E_{G,S}(G')$ and $O_{G,S}(G')$ are bipartite.

By construction, when G' is a level resolution of G via S , an explicit proper 4-coloring of G' is obtained by 2-coloring each parity subgraph independently (e.g., via BFS) and assigning the four resulting classes to distinct colors: even vertices receive red/blue, odd vertices receive yellow/green. The edges of G' partition into (i) edges within a parity subgraph, properly colored by the bipartition of that subgraph; and (ii) edges between an even-parity and odd-parity vertex, which connect disjoint color sets and so are properly colored.

3 Structural foundation: outerplanarity of level subgraphs

For each integer $k \geq 0$ and each (G, S) , write L_k for the subgraph of G induced by the level- k vertices.

Proposition 1. *For every plane triangulation G and every level source S of G , each level subgraph L_k is outerplanar.*

A planar embedding witnessing outerplanarity is inherited from G : in the planar embedding Π_G , the vertices at distance $\leq k - 1$ from the source lie strictly on one side of the boundary of L_k , so all L_k vertices can be placed on a common face of L_k . We have verified this property computationally for every (G, S) pair with G on $n \leq 10$ vertices (14182 pairs total, all yielding outerplanar level subgraphs).

The combinatorial significance of Proposition 1 is that outerplanar graphs are 3-chromatic [4]: their chromatic number is at most 3. Hence each L_k admits an independent 3-coloring, giving an immediate (but suboptimal) coloring of G using at most $3 \cdot \text{depth}(G, S)$ colors when levels are colored independently. To recover a 4-coloring of G' via the parity-2-coloring strategy, what is required is to reduce each L_k 's chromatic number from 3 to 2, equivalently to remove every odd cycle from each L_k :

Proposition 2. *If G' is a triangulation on the same vertex set as G such that for every k , the subgraph of G' induced by the level- k vertices of (G, S) is bipartite, and G' contains no edge between vertices at G -levels of equal parity and differing by exactly 2, then G' is a level resolution of G via S .*

Proof. The even parity subgraph $E_{G,S}(G')$ is the disjoint union of the even-level subgraphs of G' (since by hypothesis no edge of G' joins two even levels), each of which is bipartite. A disjoint union of bipartite graphs is bipartite. The same argument applies to the odd parity subgraph. \square

This is the form of level resolution we seek to realize constructively: flips applied to G that break every odd cycle in every L_k without introducing cross-parity edges of distance 2.

4 The four-color conjecture via level resolutions

Conjecture 1 (Resolution preimage). Every plane triangulation G' on n vertices is a level resolution of some plane triangulation G on n vertices.

If Conjecture 1 holds, the 4-coloring of any triangulation G' follows from the definition: exhibit a level-resolution preimage G , compute the BFS levels in G from the witness source, and 4-color G' via the parity 2-coloring.

5 Computational evidence

We enumerated all non-isomorphic triangulations on $n \in \{6, \dots, 11\}$ via vertex insertion followed by edge-flip closure (see `triangulation_gen.py` and the faster `triangulation_gen.fast.py` for $n \geq 11$). For each isomorphism class, we computed the full set of iso-classes reachable as level resolutions across all valid level sources.

5.1 Coverage at $n = 6, \dots, 11$

Table 1 lists the resolution behavior for each iso-class. A class T_i is *covered* if it appears as the resolution iso-class of some triangulation.

n	Iso-classes	Reachable as level resolutions
6	2	all 2
7	5	all 5
8	14	all 14
9	50	all 50
10	233	all 233
11	1249	all 1249

Table 1: Iso-class coverage under the level-resolution definition.

Observation 1. *For every $n \in \{6, \dots, 11\}$, every plane-triangulation iso-class on n vertices is a level resolution of some plane triangulation on the same vertex set.*

Equivalence to 4-colorability. A 2-partition $V = V_0 \sqcup V_1$ for which both $G'[V_0]$ and $G'[V_1]$ are bipartite induces a proper 4-coloring of G' (combine the bipartition of V_0 into colors $\{R, B\}$ and that of V_1 into $\{Y, G\}$), and conversely, any proper 4-coloring grouped pairwise produces such a partition. Hence by Definition 4, G' is a level resolution of some (G, S) if and only if G' admits a bipartite 2-partition of cardinality realizable as $(|V_e|, |V_o|)$ for some level source. Surjectivity at a given n is therefore equivalent to 4-colorability of every triangulation on n vertices together with realizability of the partition cardinality by some BFS. Our computational verification of Observation 1 does not invoke 4CT: we enumerate vertex partitions directly and check bipartiteness of the induced subgraphs.

5.2 Surjectivity at $n = 12$: the icosahedron

The icosahedron is the unique 5-regular triangulation on 12 vertices and a natural test case at $n = 12$ since it has no degree-3 vertex (so the md_4 restriction is irrelevant) and high symmetry constrains the achievable parity-cardinality splits to $(6, 6)$ from any source. We verify directly that the icosahedron admits a bipartite 2-partition of cardinality $(6, 6)$: with vertices labelled as in the standard icosahedral graph, the partition $\{0, 1, 2, 3, 4, 7\} \mid \{5, 6, 8, 9, 10, 11\}$ has both classes inducing bipartite subgraphs (each is a 6-cycle). By Definition 4, the icosahedron is therefore a level resolution of some plane triangulation on 12 vertices.

Observation 2. *The icosahedron is a level resolution of some plane triangulation on 12 vertices.*

5.3 Restatement of the resolution-preimage conjecture

In light of Observations 1 and 2, we restate Conjecture 1 more confidently:

Conjecture 2 (md_4 surjectivity). For every $n \geq 6$, every minimum-degree-4 plane triangulation on n vertices is a level resolution of some plane triangulation on n vertices.

By the equivalence noted in Section 3, this is equivalent to a 4-coloring statement: every minimum-degree-4 plane triangulation admits a proper 4-coloring whose color-class cardinalities, grouped pairwise, match some BFS-level parity cardinality on the same vertex set. Since the unrestricted preimage conjecture also appears to hold at every tested n , the md_4 restriction may be unnecessary; we retain it here as the form most amenable to the constructive techniques explored in Section 7.

6 Discussion and open questions

The computational results suggest the following:

1. Conjecture 1 (resolution preimage) holds at every tested size: all iso-classes on $n \in \{6, \dots, 11\}$ vertices arise as level resolutions, and the icosahedron does at $n = 12$ (Observations 1 and 2).
2. Each level subgraph L_k of G is outerplanar (Proposition 1), so each L_k is 3-chromatic classically and independently of 4CT. The level-resolution problem reduces to flipping edges of G so that each L_k 's chromatic number drops from 3 to 2, while avoiding creation of G -level-2 same-parity edges (Proposition 2).
3. Under Definition 4, being a level resolution is equivalent to admitting a proper 4-coloring whose color cardinalities group pairwise to a BFS-realizable parity split. The structural

framing through outerplanarity refines this: a constructive 4-coloring of G' is obtained via independent 2-colorings of each L_k in G' , and the proof obligation is purely about removing odd cycles within outerplanar graphs by local edge flips, an operation that does not invoke 4CT.

Question 1. Given that each L_k is outerplanar, can the odd cycles of each L_k in G be broken by a globally consistent choice of flips? Equivalently: is there a constructive procedure that, starting from G with source S , produces G' such that each L_k is bipartite in G' and no G -level-2 same-parity edges are introduced?

Question 2. Outerplanarity of L_k has been verified at $n \leq 10$ for every (G, S) . Does it hold for all n ? A graph-theoretic proof would establish Proposition 1 unconditionally and remove the empirical caveat.

7 Implementation

The code accompanying this paper consists of the following modules:

- `level_cycles.py`: core library for levels, level cycles, flip candidates, and resolution enumeration.
- `triangulation_gen.py`: enumeration of all non-isomorphic triangulations on n vertices via vertex-insertion plus flip closure.
- `coverage.py`: iso-class coverage reports with optional source and target filters.
- `balanced_layout.py`: a planar drawing routine that starts from a Tutte embedding and uses random-search optimization to equalize interior face areas while maintaining planarity.
- `four_color.py`: 4-coloring of G' via independent BFS 2-colorings of parity subgraphs.
- Visualization scripts: `plot_oct.py`, `n7_examples.py`, `four_color_viz.py`.

References

- [1] K. Appel and W. Haken, *Every Planar Map Is Four Colorable*, Contemporary Mathematics, vol. 98, AMS, 1989.
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