PLANE DIAMOND COLORING

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Abstract.

1. Definitions

Definition 1.1. Let G be a graph and let $u \in V(G)$. The *distance partition* of G from u is the partition $\{L_0, L_1, L_2, \dots\}$ of $V(G)$ obtained by breadth-first search from u:

$$L_0 = \{u\}, \qquad L_{i+1} = \{v \in V(G) \setminus (L_0 \cup \dots \cup L_i) : v \text{ is adjacent to some } w \in L_i\}.$$

Equivalently, $L_i = \{v \in V(G) : d(v,u) = i\}$, where $d(v,u)$ denotes the graph distance between v and u in G. We call each L_i the *i-th level* of the partition.

Definition 1.2. Let G be a maximal planar graph with a plane embedding, and let $\{L_0, L_1, L_2, \dots\}$ be the distance partition of G from some $u \in V(G)$. The *diamond scaffold* of G relative to u is the spanning subgraph $G^\diamond \subseteq G$ obtained by removing every edge $\{x,y\} \in E(G)$ such that $x, y \in L_i$ for some i.

2. Results

Theorem 2.1. *The diamond scaffold of any maximal planar graph G is 2-colorable.*

Proof. Let $\{L_0, L_1, L_2, \dots\}$ be the distance partition of G from the chosen vertex u, and let $G^\diamond$ be the resulting diamond scaffold. We show $G^\diamond$ is bipartite by exhibiting a proper 2-coloring.

For any edge $\{x,y\} \in E(G)$, the depths of x and y differ by at most 1: if $x \in L_i$, then prepending the edge $\{y,x\}$ to a shortest path from x to u gives a walk of length $i+1$ from y to u, so $y \in L_j$ for some $j \le i+1$, and symmetrically $i \le j+1$. Hence $|i-j| \le 1$.

By construction, $G^\diamond$ contains no edge with both endpoints in the same level L_i. Combined with the bound above, every edge of $G^\diamond$ joins some L_i to L_{i+1}. Color each vertex $v \in L_i$ by the parity of i. Every edge of $G^\diamond$ connects vertices of opposite parity, so this is a proper 2-coloring. $\square$

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